Homological Dimension Notes

**Def**: A abelian cat. $A$ is nonzero.

If $A$ has enough proj, define **projective dimension** $pd(A)$ as $\min_{n \in \mathbb{N}} \inf \{ i \geq 0 : A \rightarrow p_i \}$

$\inf$ = **injective dimension** $\text{id}(A) = \inf \{ i \geq 0 : I_i \rightarrow A \}$

More generally, if $P$ is some class of objects (e.g. free, flat, purple)

- if $A$ has enough $P$ then the $P$-**dimension** $\text{pd}(A) = \text{min}$ length of $P$ resolution.

**Rmk**: $pd(A) = 0 \Leftrightarrow A$ projective. **Rmk**: $pd(0) = -1$ useful convention.

**Ex**: 1. $A = \mathbb{Z}$-mod

<table>
<thead>
<tr>
<th>$\mathbb{Z}$</th>
<th>$\mathbb{Z}/p\mathbb{Z}$</th>
<th>$\mathbb{Q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$pd$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$id$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

2. $A = \mathbb{C}x^1/\mathbb{Z}^2$-mod

<table>
<thead>
<tr>
<th>$\mathbb{C}$</th>
<th>$\mathbb{C}x^1/\mathbb{Z}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$pd$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$id$</td>
<td>0</td>
</tr>
</tbody>
</table>

3. $A = \mathbb{C}x^1$-mod, $pd(C) = 1$ $(R = \mathbb{C}R)$ $\text{id}(C)$ not defined, not enough injectives

(Typically when studying rings one just works with $R$-Mod to ensure enough projectivity)

4. All quots in HW examples had $pd(\text{indecomp}) = \{ 0 \text{ if proj } \}$

**Lem**: If $\text{Ext}^1(P, A) = 0 \text{ V A}$, then $P$ is projective.

**Pf**: $B \rightarrow C \rightarrow 0$ WTS $\text{Hom}(P, B) \rightarrow \text{Hom}(P, C) \rightarrow 0$

Let $A$ be the kernel, have $\text{Hom}(P, B) \rightarrow \text{Hom}(P, C) \rightarrow \text{Ext}^1(P, A) \rightarrow$ exact
Lemma: A has enough proj. Fix \(d \in \mathbb{N}, A \in A \). TFAE

1. For any exact \(0 \to M \to P^{(\cdot,d)} \to P \to P^{(-1)} \to P^{0} \to A \to 0\), projective \(M\) is also projective.

2. \(\text{pd}(A) < d\)

3. \(\text{Ext}^n(A;B) = 0 \quad \forall n > d, \forall B\)

4. \(\text{Ext}^{d+1}(A;B) = 0 \quad \forall B\)

Pf: Clearly \(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4\). When \(d = 0\), all are asking if \(A\) is projective, Lemma says \(4 \Rightarrow\text{projective}\). Now, general case, \(4 \Rightarrow 1\): given sequence, dimensional reduction says \(\text{Ext}^0(A;B) = \text{Ext}^{d+1}(M;B) \quad \forall B\). So \(4 \Rightarrow 1\).

\(\text{Ext}^0(M;B) = 0 \quad \forall B \Rightarrow M\) is projective \(\Rightarrow 1\).

Rmk: Similar arguments show: if \(0 \to K \to P \to A \to 0\) then \(\text{pd}(K) = \text{pd}(A) - 1\).

Exercise: \(0 \to A \to B \to C \to 0\) then \(\text{pd}(B) = \max\{\text{pd}(A), \text{pd}(C)\}\) unless \(\text{pd}(C) = \text{pd}(A) + 1\), in which case \(\text{pd}(B) = \max\{\text{pd}(A), \text{pd}(C)\} = \text{pd}(C)\), but could be much less. \(\text{Ext}^0(C;B) = 0\) \(\text{pd}(B) = 0\), much less.

Lemma: Same as lemma above, but swap projective, \(\text{Ext}(A;B) \to \text{Ext}(B;A)\), etc.

Thm: Suppose \(A\) has enough proj. Then

\[
\sup \left\{ \text{pd}(A) \right\} = \sup \left\{ \text{id}(A) \right\} = \sup \left\{ \text{Ext}^n(A;B) \neq 0 \text{ for some } A, B \right\}
\]

\(\text{Ad} k \in \mathbb{N}\)

This is called \(\text{gldim}(A)\) global dimension.

Pf: Immediate from \(2 \Rightarrow 3\) above.

Note: \(R\) a ring, \(\text{gldim}(R) = \text{gldim}(R\text{-Mod})\), \(\text{rgldim}(R) = \text{gldim}(\text{Mod}-R)\).
Some examples we'll see:

<table>
<thead>
<tr>
<th>ring</th>
<th>( \mathbb{Z} )</th>
<th>( k \text{ field} )</th>
<th>( k[x_1,\ldots,x_n] )</th>
<th>( R[x_1,\ldots,x_n] )</th>
<th>reps of quieter</th>
<th>( O_{\mathbb{C}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>gldim</td>
<td>1</td>
<td>0</td>
<td>( n )</td>
<td>( \text{gldim}(R)+n )</td>
<td>1</td>
<td>( n(n-1) )</td>
</tr>
</tbody>
</table>

(Thinks) If \( \text{gldim}(A) = n \), wouldn't it be great to have a machine for producing \( n \)-length projective resolutions? Canonically?? Soon, for special cases.

**Rings of small hom. dim.**

Thm: \( R \) a ring then \( \text{gldim}(R) = 0 \iff R \) is semisimple \iff everything projective.

Certainly \( 1 \Rightarrow 2 \). \( 3 \) depends on how you define semisimple.

Bookly def'n: \( R \) is semisimple if all left ideals of \( R \) are summands, i.e. \( 0 \to I \to R \to E \to 0 \) splits.

Certainly \( I \) injective \iff \( I \) splits. \( I \) splits \iff \( I \) projective \iff summand of free \( R^I \).

Baer's Criterion: \( E \in \mathcal{R}-\text{Mod} \) is injective \iff \( \forall IC_R \) ideal, the inclusion \( IC_R \to R \)

induces \( \text{Hom}(I,E) \to \text{Hom}(R,E) \)

(Remk) Injectivity is a more general lifting criterion for all inclusions \( A \to B \).

Pf: \( \Rightarrow \) clear by remark. \( \Leftarrow \) Spext \( 0 \to A \to B \). By Zorn's lemma.

Ex: mod \( A_1 \), \( A \subseteq A_1 \subseteq B \), \( \frac{1}{b} \in E \). If \( A \subseteq B \) we win. If \( b \in B \setminus A_1 \), let \( I = \{ r \in R \mid rb \in A_1 \} \), an ideal. Get a map \( I \to A_1 \) by acting on \( b \), so

\[ I \to A_1 \to E \text{ extends to } R \to E. \]

Let \( A'' = \langle A_1, b \rangle \) and define \( A'' \to E \) by \( b \mapsto e(1) \). It works. \( \boxtimes \)

Cor: \( E \) injective \iff \( \text{Ext}^1(R, E) = 0 \) \( \forall IC_R \) ideal.

Cor: \( \text{gldim}(R) = \sup \{ \text{pd}(R/I) \mid I \text{ ideal} \} \) (not hard) \( \Rightarrow \) Semisimple \( \Rightarrow \) gldim = 0.
Now for free algs:

Prop: Let $R$ be a ring where $\text{prof} = \text{inf}$. Then $\text{gldim}(R) = 0$ or $\text{gldim}(R) = \infty$.

Pf: If $\text{pd}(A) = d \neq 0, \infty$ then $0 \to P^d \to P^{d-1} \to \cdots \to P^1 \to P^0 \to A \to 0$.

But $P^d$ injective, so first map splits, leading $0 \to K \to P^d(1) \to \cdots \to P^1 \to P^0 \to A \to 0$.

But $K \otimes P^{-i}$ so $K$ proj, so $\text{pd}(A) \leq 1$.

Ex: $k[x]$ is free for any field, have gp. $S$. Check $k[x] \otimes k[x] \otimes k[x] \otimes k[x]$.

Ex: $G = \mathbb{Z}/6\mathbb{Z}$

\[
G[x] = \frac{k[x]}{x^2 - 1} \simeq \frac{k[x]}{(x+1)(x-1)} = \frac{k[x]}{x+1} \otimes \frac{k[x]}{x-1} = k \otimes k \tag{5.5}
\]

Check $x = x_0$ so $k[x] \otimes \frac{k[x]}{y^2 = 0}$ has $\text{gldim} = \infty$.

Now dim=1

Def: $R$ is (left) hereditary if every (left) ideal is projective.

Ex: $R$ a PID (like $\mathbb{Z}$, $k[x]$) then hereditary since $I \subseteq R \otimes k[y] \Rightarrow R$ free.

Note: $0 \to I \to R$ does NOT split.

Thm: $R$ is hereditary $\iff \text{gldim}(R) \leq 1$ $\iff$ every submodule of free is projective.

Pf: (1) $\Rightarrow$ (2) since $\text{pd}(R/\mathfrak{m}) \leq 1$ so $I$ projective by outer lemma.

(2) $\Rightarrow$ (3) $0 \to S \to F \to C \to 0$ $\text{pd}(C) \leq 1 \Rightarrow S$ projective.

(3) $\Rightarrow$ (1) clear.

(1) $\Rightarrow$ (2) $\text{pd}(R/\mathfrak{m}) = \text{pd}(I)+1$ or $\text{pd}(I) = \text{pd}(R/\mathfrak{m}) = 0$.

Since $\text{pd}(I) = 0$, $\text{pd}(R/\mathfrak{m}) \leq 1$. $\Rightarrow \text{gldim}(R) \leq 1$.

Our big other class of examples will be your path algds.
Canonical Resolution

Space $R$ is a $k$-alg for field $k$. Let $S = \mathbb{R} \otimes R^p$. $S$-Mod = $(R, R)$-bimod.

Space we can find one resolution of $R \otimes S$ by free $S$-modules.

(These aren't easy to find !!)

Now apply $(-) \otimes_R M$ for any $R$-module $M$.

... $\rightarrow S \rightarrow S^0 \rightarrow R$

... $\rightarrow (R \otimes M) \rightarrow (R \otimes M)^0 \rightarrow M$

Observer: 1 still exact! This is because $S$ is free as a right $R$-module, so $\text{Tor}^1(S, M) = 0$, hence $- \otimes M$ will not create acyclic complex, built from $S$.

Thus we get a free resolution of $M$ !!. This is functional in $M$ since $- \otimes M$ is.

Call this a canonical resolution of $M$, given by a canonical resolution of $R$.

**Ex 1:** $R = \mathbb{C}[x]$ $S = \mathbb{C}[y, z]$ where $y = x \otimes 1$, $z = (ax (y, b) \otimes x)$

As an $S$-module, $R \cong S/(y, z)$ so $(S/(y, z)) \rightarrow R$ is con. res.

Thus for any $R$-module

$0 \rightarrow \mathbb{C}[x] \otimes M \rightarrow \mathbb{C}[x] \otimes M \rightarrow M \rightarrow 0$ is con. res.

$\Psi$ is just mult., b/c $y \rightarrow \text{act. of } x$, from $\rightarrow \text{FM}$.

$\Psi$ is mult by $y-z$, i.e. $\text{from } \rightarrow x \otimes M - \text{for } M$.

(Hence $\text{fillen}(\mathbb{C}[x]) \leq 1$.)

(Note: $\mathbb{C}[x] \otimes M$ is free $\Rightarrow M$ is fill.)
Ex 2: The useless yet universal example: the bar resolution. Every k-alg has a free bimodule resolution as follows: (work right to left)

\[ \ldots \rightarrow R \otimes R \otimes R \otimes R \rightarrow R \otimes R \otimes R \rightarrow R \rightarrow 0 \]

\[ \text{for some } f : g \rightarrow h \rightarrow g \rightarrow 0 \]

\[ d(a_0 \otimes \cdots \otimes a_{n+1}) = \sum (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \quad (\text{think: simplicial homology}) \]

Of course, this is far from being minimal or good for computing 
\text{gldim}, but you can use it for certain formal arguments.

Ex 3: A quiver. Recall \( A = k[Q] \) the path alg. \( e_i k[Q] e_j = \text{path from } i \to j \).

We'll work with projective rather than free, you'll see why it's ok.

Here's the canonical resolution:

\[
\begin{align*}
0 & \rightarrow \bigoplus_{x \in E} A e_0 e_x A \\
& \rightarrow \bigoplus_{i \in V} e_i A e_i A \\
& \rightarrow A \\
& \rightarrow 0
\end{align*}
\]

These summands are projective \( A \otimes R^P \)-modules (for the depicted \( e_0 e_i \))
and projective as right or left \( A \)-modules (many copies of \( A e_0 \) or \( e_i A \))

so \( 1 \) \text{ is still exact } \( 0 \rightarrow \text{Tor}^{\infty}(e_i A, M) = 0 \)

\( 2 \) \text{ \( A e_0 \otimes e_i M \) is still proj as left \( A \)-module. } \text{ (just dim } e_i M \text{ copies of } A e_0 \)

This our argument still works to get a counit of \( M \) for any \( A \)-module \( M \).

Thus \text{gldim}(A) = 1 \quad (\text{unless no edge, then gldim}(A) = 0 \text{ simple})

Explicitly, if \( M = e_i M \text{ then } \bigoplus_{x} A e_0 e_x M \rightarrow \bigoplus_{i} A e_0 e_i M \rightarrow M \rightarrow 0 \)
Ex. \[ 0 \to x \to y \to 0 \quad M = 0 \to C \to 0 \quad M_2 = C \quad M_1 = M_0 = 0 \]

\[ \oplus Ae \oplus M_1 = Ae \oplus M_2 = Ae_2 = 0 \to C \to 0 \to C \to 0 \]

\[ \oplus Ae \oplus M_1 = Ae_3 \oplus N_2 = Ae_3 \quad 0 \to 0 \to C \]

The map \( Ae_3 \to Ae_2 \) is composition w/ \( x \), i.e.

\[ 0 \to 0 \to C \]

Another major source of canonical resolution: Koszul complex.

This means two different things in two different contexts! Don't confuse!!!

1. Kosz. cx of regular sequence
2. Kosz. cx of positively graded algebra

I'll do this now.

Let \( S \) be a commutative ring. Interested in resolving \( S/(x_1, x_2, \ldots, x_n) \) for \( x = (x_1, x_2, \ldots, x_n) \in S \).

(Example: \( R = \mathbb{K}[x_1, x_2] / \mathbb{K}x_1 \mathbb{K}x_2 \).

Then as \( R \)-module, \( S \) is a \( R \)-module, \( S = S/(y_1, y_2, \ldots, y_n) \).

If \( n = 1 \), \( 0 \to S \to S \to S/(x) \to 0 \) is exact \( \iff \) \( x \) is a non-zero divisor.

If \( n = 2 \), \( 0 \to S \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} S \xrightarrow{\begin{bmatrix} \xi \xrightarrow{\begin{bmatrix} y \end{bmatrix}} S \to S/(x,y) \to 0 \) is exact \( \iff \) \( x \) is a non-zero divisor.

Why? \( \text{Ker}(S \xrightarrow{\begin{bmatrix} y \\ x \end{bmatrix}} S) = \{f \mid xf = 0 \} \]

\( \text{Ker} \left( S \xrightarrow{\begin{bmatrix} y \\ x \end{bmatrix}} S \right) = \{ fg \mid xg + yh = 0 \} \) in \( S/(x) \) have \( yh = 0 \)

\( \Rightarrow h = 0 \Rightarrow h = 0. \]

So \( x(g+yk) = 0 \Rightarrow g+y = 0 \Rightarrow g = -yk. \)

Thus \( (g,h) \) is image of \( k \) via \( \begin{bmatrix} y \\ x \end{bmatrix} \).
Def: The Koszul complex $Kz(S, x)$ is \( \bigoplus_{i=1}^{n} (S \xrightarrow{x_i} S) \).

We haven't done \( \otimes \) of complexes yet, so both explicit version.

In hom degree \(-i\) have \( S^{(i)} \) indexed by \( Ic \{1, \ldots, r\} \) of size \( i \).

Call this summand \( S_i \), i.e. \( \bigoplus_{i=1}^{n} S_{i} \) is in hom deg \(-i\).

The differential sends \( S_i \to \bigoplus_{j=1}^{n} S_{i-1,j} \) via mult by \( (-1)^{k} x_j \) where

\[ k = \# \{ j : j \leq i - 1 \} \cap I \]

Signs make \( d^2 = 0 \).

\[
\begin{align*}
0 & \to S_{123} \xrightarrow{x_3} S_{12} \xrightarrow{x_2} S_{1} \xrightarrow{x_1} S_{0} \to S/(x_1, x_2, x_3) \\
& \quad \\
& \end{align*}
\]

Clearly \( h^0(Kz) = S/(x_i) \), but rest of column not so clear, not usually exact!

Def: An ordered sequence \( z \) is regular if \( x_i \) is not in \( S_{i-1} \).

Thm: \( h^i(Kz(S, x)) = 0 \) \( \forall i < 0 \) if \( z \) is regular. PE: Later.

Rmks: 1) Not iff. In fact, \( Kz(S, z') \cong Kz(S, z) \) if \( z' \) a permutation of \( z \), but regularity is NOT preserved by permutation!

Ex: \( S = \langle x, y, z \rangle \) \( z = \langle x, y(1-x), z(1-x) \rangle \) regular

\( z' = \langle z(1-x), y(1-x), x \rangle \) not regular.

2) \( R \) local + noeth, or \( R \) graded in degrees \( \geq 0 \) and \( x_i \) all homogeneous w/ deg \( x_i > 0 \)

\( \implies \) permutation of regular is regular.

3) \( S = \langle y_1, \ldots, y_r, z_1, \ldots, z_r \rangle \) \( \tilde{z} = (y_1 - z_1, \ldots, y_r - z_r) \) is regular, so \( Kz \) gives canonical bundle restrict of \( R = \mathcal{O}_{\mathbb{P}^r}(-1) \). glbnd = \( r \).
\[ S = \mathbb{C}[x_1, \ldots, x_n] \quad x = (x_1, \ldots, x_n) \quad S/(x) = \mathbb{C}. \]

so get \( \mathbb{C} \to \cdots \to S \to \cdots \to \mathbb{C} \) exact.

To compute \( \text{Ext}^i(\mathbb{C}, \mathbb{C}) \) simply \( \text{Hom}(\mathbb{C}, -) \) to get

\[
\begin{pmatrix}
\mathbb{C} & \mathbb{C} \\
\mathbb{C} & \mathbb{C}
\end{pmatrix}
\]

all differentials 0. i.e.

\[
\dim \text{Ext}^i(\mathbb{C}, \mathbb{C}) = \binom{n}{i}
\]

i.e. \( \text{Ext}^i(\mathbb{C}, \mathbb{C}) \cong \Lambda^i(\mathbb{C}) \).

soon we'll see that \( \text{Ext}^*_S(\mathbb{C}, \mathbb{C}) \) is a graded algebra, and thus a hom of graded algebras.

Baby example of Koszul duality blow positively graded algebras.

Note: \( \text{Ext}^*_\Lambda(\mathbb{C}, \mathbb{C}) \cong S \) as graded algebras.