Tracy Cots: Recall some defns from day 1.

**Def:** $K(A)$ the homotopy category
- an additive cat.
- It is a graded cat, equipped w/ a grading shift functor (or translation functor).

\[ A[i] = A^{\oplus i} \quad \text{and} \quad d_A[i] = -d_A \]

$sA \rightarrow B$ then $A[i] \rightarrow B[i]$ is the same underlying map (no sign).

**Prop:** In $K(A)$, any h.c. is an epi. Moreover, the quotient functor

\[ \text{Ch}(A) \rightarrow K(A) \]

is universal w'th this property. **Pf:** See Weibel, for cylinder exercise.

Recall $K(A)$ is still additive, but even when $A$ is abelian, $K(A)$ is rarely abelian.

\[ 0 \rightarrow B \xrightarrow{\text{cone}(f)} A[i] \rightarrow 0 \]

$\mu = \ker v$ but $\mu$ is not monic.

\[ A \xrightarrow{f} B \xrightarrow{u} \text{cone}(f) \]

$uf = 0$ but $f \neq 0$

In stead $K(A)$ is triangulated.

**Def:** Let $K$ be an additive graded cat. A **triangle** in $K$ is the data

\[ A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[i] \]

which also leads to the data.

\[ \cdots \rightarrow \text{cone}(h) \rightarrow A \rightarrow B \rightarrow C \rightarrow A[i] \rightarrow \cdots \]

A *morphism of triangles* is

\[ A \rightarrow B \rightarrow C \rightarrow A[i] \]

\[ X \rightarrow Y \rightarrow Z \rightarrow X[i] \]

all squares commute.

**Def:** A **triangulated cat** is an additive graded cat $K$ equipped with a collection of special triangles called distinguished triangles, satisfying axioms $\text{TRO-TR4}$ below.

To motivate these axioms:

**Thm:** $K(A)$ is triangulated, when $\Delta = \{ \text{triangles isomorphic in } K(A) \text{ to}

\[ A \xrightarrow{f} B \xrightarrow{u} \text{cone}(f) \xrightarrow{v} A[i] \]

for some cone map $f$.
Remark: \[ 0 \to \frac{p_2}{b_2} \to \frac{p_3}{b_3} \to \frac{p_1}{b_1} \to 0 \] is a s.e.s. of complexes, but does not give rise to a d.t. in \( K(\mathbb{Z} mod) \)!

only degree-split s.e.s. does! Recall our example for every s.e.s.
\[ 0 \to P \xrightarrow{\text{incl}} X \xrightarrow{d} 0 \] when \( X^i = P \otimes_{\mathbb{Q}} k^i \) is a cone of \( P \to X \to Q \to 0 \).

Axioms: (TRO) 1) \[ \Delta \text{ is closed under } \equiv \text{ of triangles.} \]
2) \( (A \to B \to 0 \to A[1]) \in \Delta \text{ for all } A \in C(\mathbb{Z}) \)

Pf for \( K(\mathbb{A}) \): 1 by def. 2 b/c Cond. (ii) \( \Longleftrightarrow \)

(TRY) Any map \( A \to B \) extends to some \( D \in (A \to B \to C \to A[1]) \in \Delta \).

Remark: Other axioms will ensure \( C \) is well defined up to isom, but NOT up to more than.

Some problem we noticed for cones earlier. In general, \( C \) is called a cone of \( f \).

Pf for \( K(\mathbb{A}) \): Take the cone.

With \( \boxplus \) the next axiom on error term:

(TRA) given the triangle \( \Delta = (A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]) \), define
\[ \text{rot}(\Delta) = (B \xrightarrow{\text{def}} C \xrightarrow{h} A[1] \xrightarrow{id} B[1]) \]
and \( \text{rot}^\ast(\Delta) = (A \xrightarrow{id} A \xrightarrow{id} B \xrightarrow{g} C) \).

Then \( \Delta \in \Delta \Rightarrow \text{rot}^\ast(\Delta) \in \Delta \).

Pf for \( K(\mathbb{A}) \) - w/o correction: WTS \( B^u \to \text{Cone}(f) \to A[1] \to B[1] \) is in \( \Delta \)

\[ \begin{array}{cccc}
B_1 & \to & B_2 & \to \\
\uparrow & & \uparrow & \\
A_1 & \to & A_2 & \to
\end{array} \]

\[ \text{Cone}(A) = \left( \begin{array}{ccc}
B_1 & \to & B_2 \\
\downarrow & & \downarrow \\
A_1 & \to & A_2
\end{array} \right) \sim A[1] \]

but the map \( A[1] \to \text{Cone}(A) \) is NOT "inclusion", not a clean map.
We discuss how E.S. affects differences but not chain maps.

The actual chain map $\mathbb{A}[1] \rightarrow \mathbb{C}[1]$ is

$$a \rightarrow (a, 0, -fa)$$

This $\mathbb{A}[1] \rightarrow \mathbb{C}[1] \rightarrow \mathbb{B}[1]$ is $\mathbb{F}[1]$, not $\mathbb{F}[1] !$

But this $\mathbb{D}[1] \rightarrow \mathbb{A}[1] \rightarrow \mathbb{C}[1]$ is not $\mathbb{F}[1]$!!

But it is homotopic to $\mathbb{F}[1]$. When,

(TR3) Given $\xymatrix{A \ar[r] & B \ar[r] & C \ar[r] & A[1] \ar[r] & \Delta}$

$$\omega: f_1 \Rightarrow f_2$$

$$X \rightarrow Y \rightarrow Z \rightarrow X[1] \rightarrow \Delta$$

\[ \exists \theta \] (but not nec. unique)

**Proof:** Easy exercise will cancel

for $\mathbb{K}[a]$.

Before getting to the core axiom (TR4), some consequences:

1. $\xymatrix{A \ar[r] & B \ar[r] & C \ar[r] & A[1] \ar[r] & \Delta} \Rightarrow gf = 0$

\[ \text{PF:} \quad \xymatrix{A \ar[r] & A \ar[r] & 0 \ar[r] & A} \]

\[ \exists f, \exists \lambda \ar[r] & \exists h \ar[r] & A \]

2. **Def:** A functor $K \xrightarrow{h} \mathbb{B}$ is a **homological functor** if

\[ \Rightarrow h(\mathbb{C}[1]) \rightarrow h(\mathbb{A}) \rightarrow h(\mathbb{B}) \rightarrow h(\mathbb{C}) \rightarrow h(\mathbb{A}[1]) \rightarrow h(\mathbb{C}[1]) \rightarrow \cdots \] is exact.

**Prop:** If $h$ is a cochain, $h: \mathbb{K}[a] \rightarrow \mathbb{B}$ is a homological functor.

**Rmk:** $h$ is additive then might not have built-in hom functor on $\mathbb{K}[a]$.

**Thm:** For any $\mathbb{K}[a]$, $\text{Hom}_K(X, -)$ is a homological functor to $\mathbb{Z}$-mod.

**PF:** ETS

$\text{Hom}(X[A]) \xrightarrow{f_0} \text{Hom}(X[B]) \xrightarrow{g_0} \text{Hom}(X[C])$ exact in middle, then rotate to get exact copies,

Thus competition is zero by 1. If $y: X \rightarrow B$ and $gf = 0$ then $\exists f: X \rightarrow A$

as follows
\[ X \rightarrow Y \rightarrow Z \rightarrow X[17] \quad \text{(thus TR3 for the rotated triangle)} \]
\[ A \rightarrow B \rightarrow C \rightarrow X[17] \]

so \( \phi \) is folk. \[ \square \]

3. Many correlated

Corl (5-learn)

\[ X \rightarrow Y \rightarrow Z \rightarrow X[17] \]

\[ X' \rightarrow Y' \rightarrow Z' \rightarrow X'[17] \]

Corl: \( X \sim Y \) then \( Z \sim 0 \)

Corl: \( X \sim Y \rightarrow Z \sim X[17] \) then \( w = 0 \Rightarrow \triangle \) trapezoid is isomorphic to \( (K^\circ \rightarrow X_2 \rightarrow X) \).

and such triangles are always distinguished.

Now for TR4

3rd isom theorem: \( C \hat{\circ} B \hat{\circ} A \) makes the \( A / B \cong (A \hat{\circ} C) / (C \hat{\circ} B) \)

\[ o \rightarrow B \hat{\circ} A \rightarrow A / B \rightarrow o \quad \text{think of these as d.t.} \]
\[ o \rightarrow C \hat{\circ} A \rightarrow A \rightarrow o \]
\[ o \rightarrow C \hat{\circ} B \rightarrow B \rightarrow o \]

(\text{TR4}) Given \( X \sim Y \) and \( Y \sim Z \) (not part of one triangle)

then \( \exists \) d.t. \( C \hat{\circ} \rightarrow C \hat{\circ} X \rightarrow C \hat{\circ} Y \rightarrow C \hat{\circ} Z \) \( (\text{when } C \hat{\circ} \text{ any, case of } \alpha) \)

S.t. \( a \)

\[ X[17] = X[17] \]

\[ C[17] \rightarrow C[a] \rightarrow C[17] \]

AND \( b \)

\[ Z \hat{\circ} \circ V \rightarrow Z \hat{\circ} E \]

\[ C \hat{\circ} \rightarrow C \hat{\circ} E \rightarrow C \hat{\circ} \]

\[ X = X \]

\[ C \hat{\circ} \rightarrow C \hat{\circ} X \rightarrow C \hat{\circ} \]

the rest are triangles. Try going in 3D! octahedron axiom.
Yikes!! Only way to appreciate (TR4) is to use it. Really necessary.

When proving facts about t-structure (soon), OR just move on w/ your life.

Rmk: Two main examples of tri. cat: 1) K(A)
2) Stable modules over a Frobenius algebra A

Obs: A-mod
Mor: A-mod maps / morphism factor these projections (injective)

Think: Killing projections is like killing contractible complexes (exercises).

(2) is great b/c a good way to appreciate axiom is to use them in an unfamilar context.

(3) Derived categories. Next in line.

Aside: The triangulated Grothendieck gr [K] is $\mathbb{Z}\langle [M]/\text{mod}(\text{A})\rangle$.

$[A]+[C]=[B]$ when $A\to B\to C\to 0$.

$[A][I]=-\sigma A$ since $A\to 0\to \sigma A[I]$ is a tilt.

Note: $[A][I]=-\sigma A$ since $A\to 0\to \sigma A[I]$ is a tilt.

Exerc: $[K^b(\text{Vect}_{\text{fd}})]\cong \mathbb{Z}$.

Hint! Build a complex on an iterated cone, degree by degree.

Def: Let $A$ be an abelian cat. The derived cat of $A$, $D(A)$, is obtained from $K(A)$ by inverting all quasi-

Two things need to be done, quite separate: 1) make sense of this 2) how to use it.

Preview of (2): Any $M\in \text{mod} A \implies 0\to M\to 0$ in $K(A)$ is quasi to a proj resolution $P^\bullet$.

In fact, any bold above complex $M^\bullet$ is quasi to a complex of projectives, called
the Cartan - Glesingh min (soon). So then $D(A)$ the correct formal context for making
w/ proj res: $D^{-}(A)\cong\text{add}(\text{proj} A)$.

$D^{-}(A)\cong\text{add}(\text{proj} A)$.
So to compute morphisms in $\text{D}(A)$, replace all complexes with projective replacements and then use chain maps up to homotopy. More later.

Let's get (1) out of the way. What does "morally quasi" really mean?

**Motivation:** $R$ a comm ring. $S\in\text{multiset}$: $\circ 1 \in S$ $\circ fg \in S \Rightarrow fg \in S$

Then $R[S^{-1}]$ is well defined:

$$R[S^{-1}] = \left\{ \frac{a}{f} \mid a \in R, f \in S \bigg/ \frac{a}{f} \sim \frac{b}{g} \iff \exists h \in S \text{ s.t. } h(af-bg)=0 \right\}$$

(equv., $\frac{a}{f}=0 \iff \exists g \in S$ s.t. $ag=0$)

What if $R$ is a non-comm ring?!! Two problems w/ defining $R[S^{-1}]$:

a) Need elements like $af^{-1}bgch^{-1}$... making $R[S^{-1}]$ too big!

b) $a=0$ if $\exists g \in S$ s.t. $ga=0$

or if $\exists f \in S$ s.t. $at=0$

or $-sat=0$

**Def:** A collection of morphisms $S$ in a category $G$ is a **localizing class** if

- $\circ 1 \in S$
- $fg \in S \Rightarrow fg \in S$
- For all $X$ $\forall \frac{b}{g} \in S$ $\exists \frac{a}{f} \in S$ s.t. $af=bg$

Use double edge to denote a morphism in $S$

- Is called Ore condition, says $af^{-1}=gb^{-1}$.

Hence no big work $af^{-1}gb^{-1}$ needed, words like $af^{-1}$ will span.

**Def:** If $S$ a localizing class, define $[S^{-1}]$ with $Ob=Ob(G)$

$$\text{Hom}(X,Y) = \left\{ \left\{ \frac{Z}{X}, \frac{Y}{Z} \bigg/ \sim \right\} \bigg/ \sim \right\}$$

we call them **roots**.
Equivalences:

- \( X \xrightarrow{e} Z \xrightarrow{f} Y \)

Composition:

- \( X \xrightarrow{g} Z \xrightarrow{f} Y \)

Indep. of choice up to equivalence:

- \( X \xrightarrow{e} Y \)

Verify that:

\[ X \xrightarrow{f} Z \xrightarrow{g} Y \]

\[ X \xrightarrow{g} Y \]

\[ X \xrightarrow{g} Y \]

\( \text{Indep. of choice up to equivalence.} \)

Rank:

Is \( \{ \text{roots} \} \) even a set? Roots of fixed object \( Z \) is,

but \( \text{Ob}(E) \) isn't.

Often people add one more condition or local property, \( E \) set of objects where

all roots go thru them up to equiv.  \( \text{Locally Small} \)

The (global) size of \( \text{Hom}(E, E) \) is as expected.

\( E \rightarrow E[S^1] \) is universal among functors where \( g \rightarrow \text{Isom} \),

In fact K(A) knows a lot about that \( E \) localized without gaps.

Big technical remark:

\[ 
\begin{align*}
\text{Ch}(A) & \rightarrow K(A) \rightarrow D(A) \\
& \uparrow \\
\text{Q} \quad \text{universal} & \rightarrow \text{Isom} \\
& \quad \text{it turns out also that } \text{Q} \quad \text{is universal} \\
& \quad \text{Q} \rightarrow \text{Isom} \\
& \text{So why not just } D(A) = \text{Ch}(A)[\text{Isom}^{-1}] \text{? B/c Q isom NOT a localization class} \\
& \quad \text{inside Ch}(A)!!!}
\end{align*}
\]

But ultimately, \( \text{Hom}_{\text{D}(A)}(A^\circ, B^\circ) = \) some crazy roots mod equivalence

You NEVER use this. Next step is to understand thy letter.
Now for a key application:

Then, let $K$ be triangulated, $h: K \to B$ a homotopy functor, and $Q = \{ f \mid h(f) \text{ is an isom} \}$. Then $Q$ is a localizing class and $K[Q]$ is triangulated.

Sketch: $Q$ is multiplicative.

apply $h$. Since $W(s)$ is isom, $h(s(C))$ is isom, so h is.

$\Rightarrow h(A) \to h(B) \to h(C) \to h(W(C)) \to h(B(C)) \to W(C) = 0$.

$\Rightarrow \ker u \to \ker h(x) \to \ker h(y) \to 0 \to 0 \to 0 \to \ker h(x)$.

Exercise! $sf = 0 \iff \exists t$, $ft = 0$.

Define a triangle in $K[Q]$ as the image of DT from $K[Q]$.

Warning: Erroratum in Weibel?

Eq: What is $\text{Cone}(\Delta X \to Y)$?

So let it be $A \to X \to Y \to C \to A[1]$. 