Math 607 (Homological Algebra), Fall 2021
Exercises

There will be one exercise per class. I'll include readings by the exercise.

Week 1

Reading: Crawley-Boevey's notes, p3-6. Gabriel's theorem is on p19.

T. This exercise is about embedding the category of graded modules over a graded ring into ordinary modules over an ordinary ring. Let $R = \oplus_{i\in\mathbb{Z}} R_i$ be a $\mathbb{Z}$-graded $\mathbb{C}$-algebra, and $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded R-module.

Let $q \in \mathbb{C}$ be an invertible complex number which is not a root of unity. Define an invertible operator h on M, where $hm = q^i m$ for any $m \in M_i$. Note that h is diagonalizable, and its eigenspaces are the graded pieces M_i. Now M is a module over both R and $\mathbb{C}[h, h^{-1}]$. However, it is not a module over $R[h, h^{-1}]$ since the elements of R do not commute with h.

1. Compute hrh^{-1} for $r \in R$ homogeneous.
2. Use this relation to define a new ring R_h, having R and $\mathbb{C}[h, h^{-1}]$ as subrings which generate it, where h acts by conjugation in the appropriate way. Note: This ring R_h has the same "size" as $R[h, h^{-1}]$, but I don't need you to prove this.
3. If R_h was defined correctly then there is a functor F from graded R-modules to R_h-modules, preserving the underlying R-module structure of M and letting h act as above. Is F full? Is F faithful?
4. The functor F is not essentially surjective: find an R_h-module which does not come from a graded R-module. Which R_h-modules come from graded R-modules? (No rigorous proof required, but think about how you would define an inverse functor on this category.)

T/R. (WARMUP, DO NOT WRITE UP.) For each of the categories below, please find:

- All the simple modules.
- All the indecomposable modules.
- Which indecomposable modules are projective.
- All the non-split short exact sequences whose outer terms are indecomposable.

1. R-mod for the ring $R = \mathbb{C}[x]/(x^4)$.
2. Q-rep for the quiver $\bullet \to \bullet \to \bullet$.

R. Consider the following quiver Q. It has four vertices named $1, 2, 3$ and $*$. It has three arrows, one from i to $*$ for each $i \in \{1, 2, 3\}$. (This is a particular orientation on the D_4 quiver.)

1. Find all the simples and indecomposable projectives.

2. By the Gabriel theorem, there is one isomorphism class of indecomposable object M whose dimension at vertices $1, 2, 3$ is one, and whose dimension at vertex $*$ is 2. Find M, and show that any other indecomposable of this dimension is isomorphic to M.

3. Find a short exact sequence $0 \to P \to Q \to M \to 0$ where P and Q are projective.

Week 2

Reading: Weibel Ch 1.1, 1.3, 1.4, 1.5. Parts of Ch 2.2, 2.3.

T. Find projective resolutions of:

1. All indecomposables in R-mod for the ring $R = \mathbb{C}[x]/(x^4)$. Also find injective resolutions for this example.

2. All indecomposables for Q-rep for the quiver $\bullet \to \bullet \to \bullet$.

3. The module $\mathbb{C}[x, y]/(x^2, xy, y^2)$ over the ring $\mathbb{C}[x, y]$.

T/R Warmup. Let F be the functor from $\mathbb{C}[x]/(x^4)$-modules to $\mathbb{C}[x]/(x^2)$-modules which sends M to the quotient $M/(x^2 \cdot M)$. (What does it do to morphisms?) Apply F to your projective resolutions from **2T.**, and compute the homology of the result.

R. Practice with cones, slightly different notation from class. In this problem, there is a termwise split s.e.s.

$$0 \to B^\bullet \to X^\bullet \to A^\bullet \to 0, \tag{1}$$

so that $X^i = A^i \oplus B^i$, but the differential is not diagonal.

1. (No need to write up) Confirm the statement made in class: that such complexes are in bijection with chain maps $A[-1] \to B$. For $f \colon A[-1] \to B$, let X_f denote the corresponding complex (i.e. the cone of f).

2. Classify all chain maps $\varphi \colon X_f \to X_g$ which respect the short exact sequence (1). Under what conditions are X_f and X_g isomorphic?

3. Under what conditions is (1) split, rather than just termwise-split? (Hint: use the previous part of the exercise.)

4. Classify chain maps $X_f \to X_g$ modulo homotopy! (Upshot of this exercise: I complained that cones are not canonical. They're not even canonical in the homotopy category. But they are canonical when certain morphism spaces vanish...)

5. (Optional, but recommended) Now let A, B, C be complexes. Classify all complexes X where $X^i = A^i \oplus B^i \oplus C^i$, and where the various inclusions and projections give a filtration of X with quotient C, middle B, and sub A.

Week 3

Reading: Weibel Ch 2, then MacLane Chapter 8. Weibel 1.6 on the Freyd-Mitchell embedding theorem.

T. Practice with computing Ext. The phrase "compute all exts between simples" means that for any two simple modules S and S' (possibly $S = S'$) and any $i \in \mathbb{Z}$ you need to find $\mathrm{Ext}^i(S, S')$. The answer is NOT symmetric (you can't swap S and S').

1. Most of this exercise deals with the category of $\mathbb{C}[x]$-modules. Compute all exts between finite-dimensional simples.

2. Let $M_{(\lambda)}$ denote the one-dimensional $\mathbb{C}[x]$-module with eigenvalue λ. Take some non-split short exact sequence of your choice between finite-dimensional modules which admit nonzero morphisms from $M_{(\lambda)}$. Compute the long exact sequence associated to $\mathrm{Hom}(-, M_{(\lambda)})$.

3. Compute the long exact sequence associated to $\mathrm{Hom}(M_{(\lambda)}, -)$. Hint: don't use injective resolutions!

4. Let Q be a type A_3 quiver with some orientation. Compute all exts between simples: compute $\mathrm{Ext}^i(S_j, S_k)$ for all $i \in \mathbb{Z}$ and all vertices j and k. Make a hypothesis about what happens if you change the orientation.

R. In class I described functors between representations of different quivers, which I called the *source-push* and the *sink-pop*. Consider the quiver Q_1 which is $\bullet \to \bullet \to \bullet$, and Q_2 which is $\bullet \leftarrow \bullet \to \bullet$. The source-push gives a functor F from Q_1 modules to Q_2 modules, by inverting the orientation around the first vertex.

Also, recall the setup of Gabriel's theorem. For a quiver Q with vertex set V, let $[Q]$ denote the vector space with basis $\{\alpha_i\}_{i \in V}$. For a quiver representation M, let $[M]$ or $\dim M$ denote

$$[M] := \sum_{i \in V} \dim M_i \cdot \alpha_i \in [Q].$$

Identify $[Q]$ with $[Q']$ whenever they are related by a source-push or a sink-pop.

For this A_3 Dynkin diagram, identify $[Q_j]$ with a subspace of $\mathbb{C}^4$ by setting $\alpha_i = x_i - x_{i+1}$.

1. Compute how F acts on all indecomposables. Is F exact? Is it right exact? (If not, what is the problematic s.e.s)

2. Compute the higher derived functors of F.

3. For a representation of Q_1, when the map out of the first vertex is injective, the rank-nullity theorem determines $[FM]$ from $[M]$. For which indecomposables is this map injective? For which indecomposables is there a non-trivial higher derived functor?

4. Show that the map $[M] \mapsto [FM] - [L^1 FM]$ gives an involution $[Q_1] \to [Q_2] = [Q_1]$. This involution comes from a reflection on $\mathbb{C}^4$: what reflection is it?

Week 4

Reading: MacLane Chapter 8, supplement on Gaussian Elimination.

T.

1. Prove that if g is monic (in an abelian category) then $g = \ker(\mathrm{coker}(g))$.

2. Prove that $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $gf = 0$ and for all members x of B with $g(x) = 0$, there exists a member y of A such that $f(y) = x$.

T/R warmup.

1. Here's an old one I forgot to assign: Let C be the cone of a chain map $f\colon A \to B$. Thus there is a short exact sequence of complexes
$$0 \to B \to C \to A[1] \to 0.$$
Prove that in the long exact sequence, the map $h^i(A[1]) \to h^{i+1}(B)$ agrees with the map $f_*\colon h^{i+1}(A) \to h^{i+1}(B)$.

2. Let $A = \mathbb{Z}[x]/(x^2-1)$, i.e. A is the group algebra of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$. Let S and T denote the A-modules, both isomorphic to $\mathbb{Z}$ as $\mathbb{Z}$-modules, which correspond to the sign and trivial representations.
   (a) Show that S (resp. T) is both a sub and a quotient of a free module, but is nonetheless not a summand.
   (b) Find free resolutions of S and T. What is $\mathrm{Ext}^i(S,S)$, $\mathrm{Ext}^i(S,T)$, $\mathrm{Ext}^i(T,S)$, and $\mathrm{Ext}^i(T,T)$.

R. This problem continues the setup of the warmup: $A = \mathbb{Z}[x]/(x^2-1)$, and S and T are the trivial and sign modules. Because A is the group algebra of $\mathbb{Z}/2\mathbb{Z}$, this means that we can take the tensor product of A-modules over $\mathbb{Z}$. We write $\otimes$ for $\otimes_{\mathbb{Z}}$. Explicitly, x acts on $M \otimes N$ as $x \otimes x$. For example, T is the monoidal identity, and $S \otimes S \cong T$.

1. Find an explicit decomposition $A \otimes A \cong A \oplus A$.
2. Let F denote the two-term complex $(\underline{A} \to T)$. The differential is the quotient map, thinking of T as $A/(x-1)$. Use Gaussian elimination to compute a minimal complex for the two-term complex $F \otimes A$.
3. Let G denote the two-term complex $(T \to \underline{A})$. The differential sends $1 \in T$ to $(x+1) \in A$. Use Gaussian elimination to prove that $F \otimes G \simeq \underline{T}$, the monoidal identity in complexes of A-modules. Hence $F \otimes (-)$ and $G \otimes (-)$ are inverse functors on the homotopy category!
4. Compute a minimal complex for $F \otimes F$ (remember, when taking tensor products of complexes, some sign is introduced into the differential).
5. (If you have time) Let Λ denote the two-term complex $(\underline{A} \to A)$ where the differential is multiplication by $(x-1)$. Compute the minimal complex of $F \otimes \Lambda$. Be careful!

Extra credit! The example from **4R.** is near and dear to my heart, and is a toy model for certain important complexes in the categorified braid group. Let's explore deeper. The theme will be one of *categorified eigenvectors*. Note that $F \otimes \underline{A} \simeq \underline{A}$, so that F acts on $\underline{A}$ just like the *scalar functor* which is the monoidal identity. We call $\underline{A}$ a *weak eigenobject* for F. Note also that $(F \otimes F) \otimes \underline{A} \simeq \underline{A}$.

1. Compute the morphism space in the homotopy category from the one-term complex T (in any homological degree) to F. Compute the morphism space from any shift of T to $F \otimes F$.

2. Find a chain map $\alpha\colon \underline{T} \to F \otimes F$ for which $\alpha \otimes \mathrm{id}_A$ is the homotopy equivalence $\underline{A} \to (F \otimes F) \otimes \underline{A}$. In this sense, $\underline{A}$ is a *(strong) eigenobject* for $F \otimes F$ with *eigenmap* α.

3. Confirm that there is no morphism $\underline{T} \to F$ which induces the homotopy equivalence $\underline{A} \to F \otimes \underline{A}$.

4. Let Λ_α denote the cone of α. What is $\Lambda_\alpha \otimes A$? Hint: In general, $\mathrm{Cone}(f) \otimes M \cong \mathrm{Cone}(f \otimes \mathrm{id}_M)$.

5. Let β denote the nontrivial morphism $T[-2] \to F \otimes F$ you computed. Show that β induces a homotopy equivalence from $\Lambda_\alpha[-2]$ to $(F \otimes F) \otimes \Lambda_\alpha$, making Λ_α an eigenobject with eigenmap β.

6. Let Λ_β denote the cone of β. What is $\Lambda_\alpha \otimes \Lambda_\beta$?

You should think of (4) as categorifying the statement that, whenever m is an eigenvector for z with eigenvalue κ, then $(z - \kappa)m = 0$. Then (6) categorifies $(z - \kappa_1)(z - \kappa_2) = 0$. The most interesting thing is that one of your free resolutions from the warmup categorifies the projector $p_1 = \frac{z-\kappa_2}{\kappa_1-\kappa_2}$, projecting to the κ_1-eigenspace. What categorifies p_2?

Week 5

Reading: Weibel 3.4 (there are some errors in this chapter, or so I heard), Weibel flat resolution lemma in 3.2 (need to read some 3.1 to understand), then start homological dimension in Ch 4.

T. Prove that a k-extension $[X]$ corresponds to $0 \in \mathrm{Ext}^k(A, B)$ if and only if it is equivalent to a split k-extension.

T. challenge Let $R = \mathbb{C}[x]/x^k$. We have already computed that $\mathrm{Ext}^i(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ for each $i \geq 0$. Compute the ring structure on $\mathrm{Ext}^*(\mathbb{C}, \mathbb{C})$. (This depends on k, in a surprising way!!) For each k, find a 2-extension representing a non-trivial element of $\mathrm{Ext}^2(\mathbb{C}, \mathbb{C})$.

T/R Warmup. (pd is projective dimension.) If $0 \to A \to B \to C \to 0$ is exact, and $pd(C) \neq pd(A) + 1$, show that $pd(B) = \max\{pd(A), pd(C)\}$. If $pd(C) = pd(A) + 1$, find an example when $pd(B)$ is much smaller than $pd(C)$.

R.

1. Let $R = \oplus_{i \in \mathbb{Z}} R^i$ be a graded commutative $\Bbbk$-algebra (for a field $\Bbbk$), where $R^i = 0$ for $i < 0$, and $R^0 = \Bbbk$. Assume that R is finite dimensional overall; in particular, it is bounded above. Sketch an argument that R has infinite global dimension. (Hint: Use the fact that projective modules over a local ring are free. Resolve the one-dimensional representation.) For extra credit, make it rigorous.

2. Find the canonical resolutions of the following objects.

   (a) Some indecomposable quiver representations (give me two, make them at least slightly interesting).

   (b) The representation $\mathbb{C}[x, y]/(x^2, xy, y^2)$ of the ring $\mathbb{C}[x, y]$.

Week 6

Reading: Hmmm...

I had two ideas for HW today, so pick one.

T. Let $R = \mathbb{C}[x_1, x_2]$ be a polynomial ring, and $S = \mathbb{C}[y_1, y_2, z_1, z_2] \cong R \otimes_{\mathbb{C}} R$, with $y_j = x_j \otimes 1$ and $z_j = 1 \otimes x_j$. We are interested in taking the Hochschild homology HH_i of various (graded) R-bimodules; recall that these are the higher derived functors of $R \otimes_S (-)$, from S-modules to R-modules.

1. Compute HH_i of the R-bimodule R, for all i.
2. Compute HH_i of the R-bimodule B, for all i. Here, $B = S/(y_1 - z_1, y_2^2 - z_2^2)$.
3. (Extra credit) Let $f \colon B \to R$ be the R-bimodule map with $f(y_1) = f(z_1) = x_1$ and $f(y_2) = f(z_2) = x_2$. Compute the map induced by f from $HH_i(B)$ to $HH_i(R)$ for all i.
4. (Extra credit) For each i, one has a two-term complex $HH_i(B) \to HH_i(R)$. Compute its cohomology in each degree. You've just computed the triply graded homology of the unknot with a twist.

T. In Weibel, Proposition 3.2.4 it is proven that a left R-module B is flat if and only if $\mathrm{Tor}^1_R(I \backslash R, B) = 0$ for every right ideal $I \subset R$. The proof is similar to Baer's criterion. (Remember, $\mathrm{Tor}^i_R(M, N)$ takes a right R-module M and a left R-module N to an abelian group, just like how $M \otimes_R N$ is just an abelian group.)

1. When $I = (r)$ is principal, and r is a nonzerodivisor, compute $\mathrm{Tor}^i(I \backslash R, B)$.
2. When R is a PID, prove that B is flat if and only if it is torsion-free (meaning: if $m \in B$ is nonzero and $rm = 0$ for $r \in R$, then $r = 0$).
3. Let $R = \mathbb{C}[x, y]$, which is not a PID. Let $B = (x, y)$ be the ideal of x and y. Compute $\mathrm{Tor}^i(B, \mathbb{C})$. Is B torsion-free? Is B flat?

R. For the double complex attached to a chain map $f \colon A \to B$ of complexes, whose total complex is $\mathrm{Cone}(f)$, compute both spectral sequences (horizontal first, vertical first), and explicitly exhibit the filtration on $h^k(\mathrm{Cone}(f))$ coming from each one. This involves constructing a short exact sequence and a long exact sequence which was mentioned in class.

Week 7

Reading: Weibel Chapter 5

T. Here is a double complex $X^{\bullet\bullet}$ of $\mathbb{Z}$-modules. The only nonzero entries are $X^{10} \cong X^{11} \cong X^{01} \cong X^{02} \cong \mathbb{Z}$. The horizontal differential is multiplication by 12 from X^{01} to X^{11}. The vertical differential is multiplication by 2 from X^{01} to X^{02}, and multiplicaiton by 8 from X^{10} to X^{11}.

1. Compute the total complex and its cohomology. (Hint: For abstract reasons, this will be an abelian group of order 16. Which abelian group is it? Does it have an element of order 16? Of order 8?)
2. Compute the spectral sequence where one takes vertical cohomology first. What filtration on the total complex do you get?

3. Compute the spectral sequence where one takes horizontal cohomology first. What filtration on the total complex do you get?

Week 8

Reading: Weibel Ch 10, Kiehl-Weissauer.

T. Given a distinguished triangle $X \to Y \to Z \to X[1]$ with maps (u, v, w), prove that $w = 0$ if and only if $Y \cong X \oplus Z$, making u and v commute with the inclusion and projection maps. Moreover, prove that for any X and Z, the triangle $X \to (X \oplus Z) \to Z \to X[1]$ is distinguished.

T/R warmup. Give a sketch for why the direct sum of two distinguished triangles is distinguished. Also, verify the rotation axiom for $\mathcal{K}(\mathcal{A})$: for $f\colon A \to B$, explicitly construct the isomorphism of triangles between the rotation of the cone sequence of $\mathrm{Cone}(f)$, and the cone sequence $\mathrm{Cone}(B \to \mathrm{Cone}(f))$.

Choose one below.

R1. Let $F\colon \mathcal{K} \to \mathcal{K}'$ be a triangulated functor (i.e. an additive functor sending distinguished triangles to distinguished triangles), and Q be the class of morphisms which is sent to isomorphisms. Prove that

1. Given any $s\colon A \to B$ with $s \in Q$, and $f\colon Y \to B$ any morphism, one can extend this data to a commutative square $ft = sg$, with $t \in Q$.
2. Given $f\colon A \to B$, the existence of some $s \in Q$ with $sf = 0$ implies the existence of some $t \in Q$ with $ft = 0$.

Hint: You do not need the octahedron axiom. For both, take the cone of s, and apply the functor F. For the second problem, apply $\mathrm{Hom}(A, -)$ to the cone of s.

R2. The source-push gives a functor F from $Q_1-\mathrm{Rep}$ to $Q_2-\mathrm{Rep}$, by inverting the orientation around the first vertex, and the sink-pop functor G gives a functor back. Remember that F is right exact, while G is left exact.

1. You have already computed the higher derived functors of F on all simples, so remind yourself. Same with G. You also found projective and injective resolutions of everything.
2. It is rarely true that a complex is quasi-isomorphic to its cohomology (viewed as a complex with zero differential). However, show that this is always true when the cohomology is concentrated in a single degree, which is the first or the last degree in the support of the complex.
3. Compute the (total) derived functors $\mathbb{L}F(S)$ and $\mathbb{R}G(T)$ for each simple quiver representation S of Q_1 and T of Q_2. E.g., find a nice complex which represents the quasi-isomorphism class of $\mathbb{L}F(S)$.
4. Compute that $\mathbb{L}F(\mathbb{R}G(T)) \cong T$ and $\mathbb{R}G(\mathbb{L}F(S)) \cong S$.
5. Deduce that $\mathbb{L}F$ and $\mathbb{R}G$ are inverse functors between $D^b(Q_1-\mathrm{Rep})$ and $D^b(Q_2-\mathrm{Rep})$. (You need to prove this for all objects of the derived category, not just for objects in the abelian category! So use the 5-lemma.)

Week 9

Reading: Weibel Ch 10, Kiehl-Weissauer.

T. Consider modules over $\Lambda = \mathbb{C}[x]/x^2$. Let M be the complex $\Lambda \to \underline{\Lambda}$, where the differential is multiplication by x.

1. Compute the hom complex $\underline{\mathrm{Hom}}(\mathbb{C}, M)$ and its cohomology.
2. Compute the hom complex $\underline{\mathrm{Hom}}(P, M)$ and its cohomology, where P is the projective resolution of $\mathbb{C}$. (Hint: This complex should mostly consist of 4D vector spaces.)
3. Is there a reason why these two cohomologies agreed?

T Challenge. Consider modules over $R = \mathbb{C}[x]$. Let M be the complex $R \to \underline{R}$ where the differential is multiplication by x.

1. Compute the hom complex $\underline{\mathrm{Hom}}(M, M)$ (it should be rank 4 over R) and its cohomology.
2. Explicitly indicate the algebra structure on $\underline{\mathrm{Hom}}(M, M)$.
3. Viewing the cohomology as a dg-algebra with zero differential, find an algebra morphism from it to $\underline{\mathrm{Hom}}(M, M)$ which is also a quasi-isomorphism. (Note: it need not be a morphism of complexes of R-modules.)

Week 10

Reading: Weibel Ch 10, Kiehl-Weissauer, Khovanov's Hopfological algebra paper.

T. Let $(D^{\leq 0}, D^{\geq 0})$ be a t-structure on a triangulated category $\mathcal{D}$. Let M and M' be two objects, and consider distinguished triangles

$$X \to M \to Y \to X[1],$$
$$X' \to M' \to Y' \to X'[1],$$

with $X, X' \in \mathcal{D}^{\leq 0}$ and $Y, Y' \in \mathcal{D}^{\geq 1}$. Prove that any map $M \to M'$ induces a unique map $X \to X'$ and a unique map $Y \to Y'$. Prove that this assignment is functorial (i.e. for a composition $M \to M' \to M''$, the maps $X \to X' \to X''$ compose to be the map $X \to X''$ induced by the composition $M \to M''$).

R. Prove that the triangulated Grothendieck group $[\mathcal{K}^b(\mathrm{vect})]$ of the bounded homotopy category of vector spaces is $\mathbb{Z}$, where the symbol of a complex is given by its Euler characteristic. This statement is false when vect is replaced with an non-semisimple abelian category $\mathcal{A}$. Prove that there is an Euler characteristic map $[\mathcal{K}^b(\mathcal{A})] \to [\mathcal{A}]$. Prove that when $\mathcal{A}$ has finite homological dimension, then there is a section $[\mathcal{A}] \to [\mathcal{K}^b(\mathcal{A})]$. Hint: the section is NOT $[M] \mapsto [0 \to M \to 0]$. Why is this not well defined!? Find a counterexample.

Week 11

Reading: Khovanov's Hopfological algebra paper.

R. Prove that any commutative square in the stable module category over a f.d. Hopf algebra extends to a morphism between distinguished triangles. (Hint: You can use any injective module you want to define the triangles, so choose the injective module through which something is forced to factor...)