Properties: 1) W generated by \( \beta \in \Delta \), hence \( \beta \Delta = \beta \) simple reflections.

2) Def: An expression for \( w \) in \( W \) is \( w = s_{i_1} s_{i_2} \cdots s_{i_d} \), \( s_i \in S \) (they exist).

Minimal length expression = reduced expression = rank. \( l(w) = d \) for rank.

Def: \( n(w) = \# \text{pos roots sent to neg roots} = \# \text{pos roots sent to neg roots} - \# \text{neg roots sent to pos roots} \).

Prop: \( l(w) = n(w) \).

Ex: \( n(4) = 0 \iff 4 \in \Delta \) \( n(3) = 1 \iff 3 \in \Delta \)

\( \beta \Delta \) means, \( \begin{cases} 
\beta_1, & \text{if } \beta_1 \in \Delta \\
\beta_2, & \text{if } \beta_2 \in \Delta \\
\beta_3, & \text{if } \beta_3 \in \Delta \\
\beta_4, & \text{if } \beta_4 \in \Delta \\
\beta_5, & \text{if } \beta_5 \in \Delta \\
\beta_6, & \text{if } \beta_6 \in \Delta \\
\beta_7, & \text{if } \beta_7 \in \Delta \\
\end{cases} \)

A rank tells you which pos roots sent to neg roots in which order. Once they go neg, they don't come back.

\{\text{ex} \in \Delta \} \leftarrow \{ \text{certain admissible orders on } \beta \Delta \} \leftarrow \{ \text{certain admissible orders on neg roots} \}

Ex: \( s_3 s_1 s_3 \) vs. \( s_1 s_3 s_1 \) \( s_5 \), \( s_3 s_1 s_3 \) vs. \( s_3 s_1 s_3 \)

3) \( W \) acts freely on \( \{ \text{bases} \} \). Pf: If \( w \neq 1 \) then \( w = s_{i_1} s_{i_2} \cdots s_{i_d} \) and \( \beta_1 \Delta \rightarrow \beta_2 \Delta \)

\( \Rightarrow \) \( W \) is free \( \Rightarrow \) \( W \) finite.

4) \( \exists! \) longest word \( w_0 \), \( w_0 \Delta = -\Delta \), \( l(w) = n(w) = \# \beta \Delta \).

\( w_0 \beta \Delta = \beta \Delta \) if \( \text{refl } w_0 = \beta \) \( w \rightarrow -\text{refl } \).

Note that \( w_0 \) need not be \( -I \in \Delta \).

\( \beta \Delta \) or \( \beta \Delta \) even/odd then \( \det w_0 = \pm 1 \).

If \( \det w_0 = \pm 1 \) then \( \det ( -I ) = \pm 1 \). Needs \( w_0 \) match up. Even when \( \Delta \) doesn't.

Ex: \( \Delta \) and \( w_0 = 8 \)

\( \epsilon_1 - \epsilon_2 \rightarrow \epsilon_{n-1} - \epsilon_n = - ( \epsilon_{n-1} - \epsilon_n ) \)

Ex: \( B_n \) \( w_0 = -I \).

\(-w_0 = \epsilon \) where \( \epsilon \) is Dynkin diagram automorphism. When \( \epsilon = -1 \), \( w = -I \).

Ex: \( \Delta \) or \( \beta \Delta \) where \( \epsilon \) is Dynkin diagram automorphism. When \( \epsilon = -1 \), \( w = -I \).
5) \( W \) has a presentation \( W = \langle s_0 s_1 s_2 s_3 \mid s_0^2 = 1, \quad \text{braid reln} \rangle \)

where \( M_{s_0 s_1} = \begin{cases} 2 \quad \theta = 90^\circ \text{ A}_1 \\
3 \quad \theta = 120^\circ \text{ A}_2 \\
4 \quad \theta = 135^\circ \text{ B}_2 \\
6 \quad \theta = 150^\circ \text{ G}_2 \end{cases} \)

1.e. \( W \) is a Coxeter group.

Rmk: \( W \) is crystallographic, i.e. \( W \) preserves a lattice e.g. \( \Lambda_{ct} = \mathbb{Z} \cdot \mathbb{O} \).

\[ \{ \text{Fin. Cryst. Cox Gps} \} = \left\{ \text{Weyl gps of root systems} \right\} \]

only very few other finite Cox gps \( \Rightarrow \) exceptional, \( H_5, H_4 \).

Ex: \( S_{n+1} = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1, \quad s_i s_j s_i = s_j s_i s_j \text{ for } |i-j| > 1 \rangle \]

\( n = 11 \quad \Rightarrow \quad \tau = \begin{bmatrix} 111111111111 \end{bmatrix} \)

- \( \ell(w) = \# \text{ cons} \text{ in a rep exp} \)
- \( n(w) = \# \text{ inversions} = \# \left\{ (i,j) \mid w(i) > w(j) \right\} \)

Ex: \( \times \times \times \times \times \)

some a \( \Phi^+ \cap w^{-1} \Phi^- \)

\( \mathbb{E}_2 \times \mathbb{E}_3 \rightarrow \mathbb{E}_4 \times \mathbb{E}_1 \) a picture is related if \( \text{NOTO STRANDS CROW TRACE} \)

Exercise: Think about type B.

6) Any two reps for \( W \) are related by braid rels (Nemoto's Thm)

(a priori could have to make longer than shorter.)

7) Suppose \( L \) semisimple with root system \( (\Phi, \Phi^+) \), and \( V \) a full \( L \)-repn

which is \( \Phi \)-adg, \( \text{Wts}(V) \subset \text{Ch}^* \). Then

a) \( \forall \lambda \in \text{Wts}(V), \quad \langle \lambda, \alpha \rangle \in \mathbb{Z} \).

b) \( \forall \alpha \in \Phi^+ \), \( W \subset \text{Ch}^* \) preserves the multiset \( \text{wts}(V) \)

Pf: \( L \)-repn gives \( sl_2 \) repn for each \( \alpha \in \Phi \). So \( S_{\alpha} \) preserves \( \text{wts}(V) \).

Jumping ahead, but need this lemma soon.
Def: Given \((\varphi, \beta)\) and \(\Delta\), the Cartan matrix is \((<\varphi, \beta>\) \). If \(\Delta\) is not the matrix of \((\varphi, \beta)\) but, since knowing \(\angle(\varphi, \beta)\) and \(\angle(\beta, \varphi)\) determines the \(\gamma\) setup, it gives length and angles, and determining \((\varphi, \beta)\) up to scalar.

Def: The Dynkin diagram is the graph w/ vertices \(\Delta\) and edges:

- \(\gamma \rightarrow\) (no edge) if \(\Theta = 90^\circ\)
- \(\gamma \rightarrow \) if \(\Theta = 120^\circ\)
- \(\gamma \rightarrow \) if \(\Theta = 150^\circ\)

Ex: \(A_n\)  
Ex: \(B_n\)  
Ex: \(G_2\)  
Ex: \(C_n\)  
Ex: \(D_n\)  
Ex: \(A_3 \times A_2\)

Theorem: 1) If \((\varphi, \beta)\) is irreducible root system then \(\Gamma\) is either (non-isomorphic list): \(A_n\), \(B_n\), \(C_n\), \(D_n\), \(E_n\), \(F_n\), \(G_2\), \(n \geq 1\).

2) Each of these do come from a root system!

Proof 1) Similar to first seminar. There we already proved i.g. \(\Gamma\) simply-laced, get paring \((\varphi, \beta)\) and NDE only pos def ones. Since paring = Cartan matrix, thus proved.

For not simply-laced - similar argument. See Humphreys. Along way, also classify.

Affine Dynkin Diagrams: when \((\varphi, \beta)\) has a 1D kernel, is pos. seconded.

To find 1D kernel: label vertices w/ integers \(\delta_i\) s.t. \(2 \delta_i = \Sigma\) neighbors w/ "oriented mult".

Proof 2) Construction. See Humphreys. \(E_6\) exercise.

Note: If you make \(E_8\) then \(E_6, E_7\) are "sub-root-systems" i.e. \(\text{Span}(\varphi, \beta)\).

Note: All non-simply-laced can be constructed by "folding" simply-laced. Exercise.
Now back to lie alg (eventually back to lie Alg)!: We know

\[ \text{[Dynkin]} \xrightarrow{\text{correspondence}} \text{[Irreducible root system]} \xrightarrow{\text{\rightleftharpoons}} \text{[Simple lie alg]} \]

We'll show \( \rightleftharpoons \)

We'll now construct \( \rightarrow \): Given \((\mathfrak{g}, \mathfrak{e}, \Delta)\) well\(\text{Con}\)struct \(\mathfrak{L}\). By gen + relns!

This will make it functional in Dynkin diagrams! Given a sub-root system get a sub-lie-alg!

Give an automorphism \(T\) of \(\mathfrak{T}\) get one of \(\mathfrak{L}\).

Recall lie alg by gen + relns. Given give a v.s. \(V = \text{Span}\{g\text{ss}\}\).

Get alg: \(\mathfrak{T}(V)\). Let \(F(V) \cap \mathfrak{T}(V)\) be subspace spanned by \(\{v_1, v_2, v_3, \ldots, v_n\}\) inside \(\mathfrak{T}(V)\).

This is closed under \([, , ]\) (by Jacobi identity; all crazy bracket reduce to this) so is a lie alg.

Free lie alg: \(\text{Relns}\): Given a subspace \(\mathfrak{R} \subset \mathfrak{F}(V)\), let \(\mathfrak{F}(\mathfrak{F}(V))\) be (algebra) ideal generated by \(\mathfrak{R}\). Then \(\mathfrak{I}(\mathfrak{R}) \cap \mathfrak{F}(V)\) is lie alg ideal, \(\mathfrak{F}(V)\) mod \(\mathfrak{I}(\mathfrak{R}) \cap \mathfrak{F}(V)\) is lie alg by gen + relns.

\(\mathfrak{T}(V) / \mathfrak{I}(\mathfrak{R}) \cong U(\mathfrak{g}(<\mathfrak{V}, \mathfrak{R}>))\). Get presentation for \(U\) too.

Ex: For \(\mathfrak{g}_3\) we had \(\mathfrak{L} = \mathfrak{h} = \mathfrak{h}_1, \mathfrak{h}_2, x_1, x_2, y, y_1, y_2, y_3\) w all commutator. This gives

\[ \text{presentation} \xrightarrow{\text{w gen}} \text{and relns} \xrightarrow{\rightleftharpoons} \text{This works for any lie alg!} \]

Ex: For \(\mathfrak{g}_3, x_3 = [x_1, x_2]\) and \(y_3 = [y_1, y_2]\) so don't need a generator.

Reln \( [h_1, x_3] = x_3(h) x_3 \) follow from \( [h, x_1] = \alpha_1(h) x_1 \) and \( [h, x_2] = \alpha_2(h) x_2 \)

Reln \( [x_3, y_3] = h_1 h_2 \) follow from other commutator.

Reln \( [x_1, x_3] = 0 \) doesn't help! \([x_1, [x_1, x_1]] = 0\) new relation? Some relations.

\( [x_2, x_3] = 0 \)

\( \mathfrak{g}_3\) has presentation:

\[ \langle x_1, h_1, y_3, x_2, h_2, y_2 \rangle \mid \begin{align*}
& [x_1, h_1, y_3, x_2, h_2, y_2] \text{ the triple, \(\mathfrak{g}_3\) alg w special basis graph} \\
& [x_1, y_3] = 0 \text{ and some relns}
\end{align*} \]

This presentation is "exposed" in the Dynkin diagram! w \(\mathfrak{g}_3\) - no interest.
Key point 1: If $\alpha, \beta \in \Delta$ then a string from $\beta$ has length $<\beta \alpha>$

\[ \Rightarrow \begin{array}{c}
\frac{\text{Ex} \alpha \text{Div} \text{Ex} \alpha \cdots \text{Ex}_x k \text{I} \text{I}}{\text{I} \text{I} \text{I} \text{I}} = 0 \\
\text{Y/L no root space}
\end{array} \]

\[ \Rightarrow \begin{array}{c}
-<\beta \alpha> + 1 \text{ times} \\
\text{Serre relation:}
\end{array} \]

Minor points: 1. $\mathfrak{sl}_2$ triples for $\beta \in \Delta$ do generate all of $\mathfrak{g}$. Just $x$'s quantum \( \mathfrak{h}^+ \).

Span $\alpha \in \mathfrak{h}^+$, $\alpha = \sum c^\beta \beta$, $c^\beta \in \mathbb{Z}_{\geq 0}$. Then can choose a path

\[ \beta_0 = 0, \beta_1 = \beta, \beta_2 = \beta_1 + \beta_2, \ldots, \alpha_n = \alpha \]

where each one is a root! (h/c of root ring theory)

\[ \Rightarrow \text{Ex} \beta_1 \text{Ex} \beta_2 \cdots \text{Ex} \beta_n \text{Ex} \alpha = x_0 \text{ up to scalar} \]

(this is why we know $[\mathfrak{L}, [\mathfrak{L}, [\mathfrak{L}]]] = [\mathfrak{L}, [\mathfrak{L}, [\mathfrak{L}]]]_7$ when $\gamma \delta \delta$ are roots.

Key point 2: $L = \mathfrak{L}^+ \oplus \mathfrak{L}^-$ is a U. There are subalgebras on $\mathfrak{h}^+ \mathfrak{h}^-$ (guaranteed by Serre), \( \gamma \delta \delta \Delta \).

\[ \begin{array}{c}
\text{Triangle decomposition:} \\
L^+ = \oplus \mathfrak{L}^+ \mathfrak{L}^- \mathfrak{L}^-
\end{array} \]

Thm (Serre): Let $L$ be the lie algebra defined by gen + rels, whose
generated by \[ \text{Ex}_x \text{Ex}_y \] \( \alpha \in \Delta \) (let $h_\alpha = [\text{Ex}_x \text{Ex}_y]$) subject to relns:

1. $[x_\alpha, h_\alpha, y_\alpha] \in \mathfrak{sl}_2$ triple
2. $[x_\alpha, y_\alpha] = 0 \quad \alpha < \beta$
3. $[h_\alpha, y_\beta] = <\beta \alpha \gamma > y_\beta$
4. $([h_\alpha, x_\beta] = [h_\beta, x_\alpha] = 0 \quad \alpha < \beta$

Then $L_\beta$ is finite and semi-simple (simple if $\gamma \Delta \Delta$). We'll build a dual \( \mathfrak{h} \) and root system \( \Phi \). If $L$ is simple, $L^\rho \cong L$. We know $L_\rho \rightarrow L$ since it generated the at least these relns.
Rank: How do we know \( L_p \neq 0 \)? If \( L \) acts w/ \( \Gamma \) then \( L_p \rightarrow L \) is ok.

Could just construct then explicitly \( \Gamma \), see Humphreys. But better way is to construct a \( L_p \)-rep, when it acts faithfully. We'll do that too.

PFA: Let \( \Gamma = \text{Lie alg by gen+rels w/o (5), } (\text{Expect to be } \infty \text{-dim.)} \)

1) \( P = P^+ + H + P^- \)
   \( P^+ \text{ gen by } X, P^- \text{ by } Y \quad H = \text{span}\{h_0\} \).
   This is just like of form of reln. If you take \( \tilde{x} = [x_1, x_2, ..., x_n, x_{n+1}] \) and act by \( [h, \cdot] \)
   you recover \( \cdot \) by \( E_y, x_j \) you get some \( h \) (which then recales), etc. Exercise.
   
   (Induction: \( E_y, x_j \) = \( E_y, x_j \) + \( E_y, x_j \) \( x_1 \) + \( E_y, x_j \) \( x_1 \) ) \( \text{unless } n = 1, \ E_y, x_j \in H \).

2) \( H \) acts semisimply, \( P \) graded by \( \text{Art.} \). \( \Rightarrow \) \( P = P^+ \oplus H \oplus P^- \).

3) More implications:
   - Any ideal is homogeneous
   - \( P \) is fd, in each degree
     - \( \dim P[x] = 1 \) for \( x \in \Delta \) (just \( \frac{1}{k!} \) no more complicated stuff)
     - \( \dim P[kx] = 0 \) if \( k \neq \pm 1 \). \( (k \in \mathbb{N}, \Rightarrow x_1 = 0) \)
   - No proper ideal of \( P \) intersects \( H \).
     - Pf: If \( \text{hom} \mathcal{O} \mathcal{H} \) have all \( \mathcal{O} \mathcal{P} \mathcal{A} \)
       with \( X(h) \neq 0 \) (some real since \( \Delta \) span) \( \Rightarrow \) have \( x_0 \neq 0 \) with \( h_0 \neq 0 \)
       since \( H \) central, eventually get all generators.
   - \( P \) has a \( ! \) maximal ideal.
     - Pf: Any proper doesn't meet \( H \). Take \text{sum of all proper. Still doesn't meet } H.
       \( \Rightarrow \) \( P \) has \( 1 \) simple quotient \( (\text{if } \Gamma \text{, non zero!}) \)

4) Constructing an action of \( P \): "Verma module".
   \( \text{Let } C = \mathcal{C} v \text{ be trivial rep of } \mathcal{D} \mathcal{O} \mathcal{H}, \text{red indec to } P, \text{ to get } V. \)
   \( \text{ie } V = U(P^+) v \text{ as vs. } (U(\mathcal{O}) = U(P^+) \otimes U(H) \otimes U(P^-) \text{ as vs.}) \)
   \( \text{ie } V \) has basis \( \{E_{y_1, y_2, ..., y_k} v\} \).
   \( \text{Action of } h \text{ is } h \cdot v = 0 \)
   
   \( \Rightarrow h \cdot y, y_k v = -e(h) y, y_k v + e(h) y, v - e(h) y, y_k v = 0, y, y_k v \).
   \( \Rightarrow V = \mathbb{R} V \mathcal{A} \mathcal{T}, \mathbb{R} \mathcal{V} \mathcal{T} v_0, \ y_k \text{ now acts by } x_k. \)
There action satisfy $123$ so $P$ does act on $V$ continuously $\implies P \neq 0$.

**Lemma:** Let $D^c P$ be given by $\mathcal{G}$ (ad $y_a^{\alpha}$) $(y_b^\beta)$. Then $D$ is a sub-

In $P$ too. (and to prove)

$\textbf{ Pf:}$ (This doesn't really use $V$ but philosophically it does)

Clearly $[L, D^-] < D^-$ since homomorphisms. If $y_a \in A$, $y_b^\alpha$ then $[x_b, y_a] = 0 = [x_b, y_a]$ in $[x_b, S] = 0$.

Now $[x_b, y_a] = 0$ so $\text{ad}_{x_b}$, $\text{ad}_{y_a}$ commute, so $[x_b, S] = (x_b y_a) \text{ (mod y_a)}$.

If $<\beta, a> \neq 0$ then $[y_a, [y_a, x_b]] = 0$

If $<\beta, a> = 0$ then $[y_a, x_b] = 0$ $\beta \perp \alpha$.

Finally $[x_a, S] = 0$. Look at $\alpha$-string of $y_b$ in $P$ (same as in $V$)

Thus to order $\hat{S}$-rep, give by

we're talked about how

we're done this condition before and exactly

we're done this condition before and exactly

and the theory for $\alpha$-string the $y_b$

we're done this condition before and exactly

So $D^-, D^+$ are proper ideals, $D^+ + D^- = (S)$ is an ideal. ETS that $L_P$ is $\mathbb{C}$ simple.

**Key Part 4:** (S) relation, all enough to make $\text{ad}_{y_a}$ ad y a nilpotent on generator $y_b$.

$\implies$ locally nilp on all of $L_P$. In $L_P$, splits into $\mathbb{F}_P$ $\mathbb{C}_2$ rep for each triplet.

$\implies$ acts $(L_P) \otimes W$.

**Key Part 5**
But we know \( \dim \text{P} = \dim \text{L} = 1 \) \( \forall \varepsilon \Delta \Rightarrow \text{true} \) \( \varepsilon \Delta \Rightarrow \). 

\( \dim \text{P} \Delta \varepsilon = 0 \) unless \( \varepsilon = \text{post- cone or nega- cone} \) (i.e. \( P^+ \) or \( P^- \)) 

but if \( \lambda \neq \kappa \Delta \) for some \( \varepsilon \) \( \lambda \) then \( \lambda \) is on a ray \( b/w \) \( \lambda \) \( \Delta \), \( \lambda \) 

Ray \( b/w \) a pos root + a neg root (not hard) so \( \lambda \) not in \( \text{extreme cone} \) and \( \dim \text{P} \Delta \varepsilon = 0 \). 

\[ \Rightarrow \dim \text{P} \Delta \varepsilon = 0. \]

So \( \dim \text{L} \Delta \varepsilon \) \( \leq \) 1 if \( \varepsilon \Delta \). 

\[ \Rightarrow \text{m is maximal, } \text{L} \Delta \varepsilon \text{ is simple.} \]

By root system. 

Proof: In similar style, Chevalley constructed algebraic groups (groups in alg geom) 

Beyond the scope of this class. 

\[ \Rightarrow \text{complex Lie groups.} \]