Def: If base field is a field, a Lie algebra is a $\mathbb{F}$-vector space $A$ with a bracket $[\cdot, \cdot] : A \times A \to A$ such that:

1. $[x, y] = -[y, x]$ (antisymmetry)
2. $[x, x] = 0$ for all $x \in A$
3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity, replacement for associativity)

Ref: If char $\mathbb{F} \neq 2$ then $[x, y] = 0$ and $[x, y] = -[y, x]$ imply each other.

In char $\mathbb{F} = 2$, they don't, need both.

Ex:
1. If any vs $[\cdot, \cdot]$ is $\mathbb{F}$-bilinear, then $A$ is a Lie algebra.
2. A $\mathbb{F}$-vector space is a Lie algebra.
3. Sub-Lie algebra is a subspace closed under bracket.

Sub-Ex: $gl(n, \mathbb{F})$ is the algebra attached to the adjoint adj. to $\mathfrak{gl}(n, \mathbb{F})$ (Lie algebra), but is NOT attached to the adjoint adj: $xy \neq yx$ for $x, y \in \mathfrak{gl}(n, \mathbb{F})$.

Swex 1: $\mathfrak{gl}(n, \mathbb{F})$ is the adjoint adj. to $\mathfrak{gl}(n, \mathbb{F})$.

Swex 2: A $\mathbb{F}$-vector space $A$ is an $\mathbb{F}$-vector space.

Def: If $\mathbb{F}$ is a field, a Lie algebra is a $\mathbb{F}$-vector space $A$ with a bracket $[\cdot, \cdot] : A \times A \to A$ such that:

1. $[x, y] = -[y, x]$ (antisymmetry)
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3. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity, replacement for associativity)

Def: A morphism of Lie algebras is a map $\phi$ such that $[\phi(x), \phi(y)] = \phi([x, y])$.

Lie $\mathbb{G} \to$ Lie $\mathbb{G}'$ is a morphism.

Lie $\mathbb{G} \to$ Lie $\mathbb{G}'$ is a morphism.
Classify the 2-sphere $S^2$ as a Lie group. Define $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C})$ to be the Lie algebra of $\text{SO}(3)$. Let $u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

If $[g, \mathfrak{g}] = 0$, then $g$ is nilpotent. Since $\text{Sp}(\mathbb{C}) = \text{U}(1) \times \text{U}(1)$, $\mathfrak{sp}(\mathbb{C}) = \mathfrak{u}(1) \times \mathfrak{u}(1)$. Thus, $\text{Sp}(\mathbb{C})$ is a Lie algebra.

Complexification of $\mathfrak{g}$ is a $\mathbb{C}$-algebra. Define $\mathfrak{g}^\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$ to be the Lie algebra of $\text{SU}(2)$. Let $\mathfrak{su}(2) \subset \mathfrak{so}_3(\mathbb{C})$ be the Lie algebra of the special orthogonal group $\text{SO}(3)$.

The space $\mathfrak{su}(2) / \mathfrak{su}(1)$ is the complexification of $\mathfrak{su}(2)$.

Example $\mathfrak{su}(2) \cong \mathfrak{sl}(2, \mathbb{C})$.

\[
\mathfrak{su}(2) = \left\{ X \in \mathfrak{gl}(2, \mathbb{C}) \mid X + X^* = 0 \right\} = \text{Sp}(i) \mathbb{R}
\]

\[
\mathfrak{sl}(2, \mathbb{C}) = \left\{ X \in \mathfrak{gl}(2, \mathbb{C}) \mid \text{Tr} X = 0 \right\} = \text{Sp}(i) \mathbb{R} \setminus \{ 0 \}
\]

Let $\mathfrak{su}(2) = \text{Sp}(i) \mathbb{R} \subset \mathfrak{su}(2)$. Then $\mathfrak{su}(2)$ is a Lie algebra.

$\mathfrak{su}(2) = \text{Sp}(i) \mathbb{R}$.

$\mathfrak{su}(2) \cong \mathfrak{sl}(2, \mathbb{C})$. Therefore, $\mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C})$.
Thm: Reps of $SL(2; C)$ are semisimple.

Pf: B/c $\pi_1(SL(2; C)) = 1$, $\text{Rep}_{C} \pi_1(SL(2; C)) \cong \text{Rep}_{C} \pi_2(SL(2; C)) = \text{Rep}_{C} (SU(2); C) = \text{Rep}_{C} (SU(2))$.

But $SU(2)$ is compact $\Rightarrow \text{Rep}_{C} (SU(2))$ is semisimple.

Def: Let $G$ be a $C^\ast$ Lie gp. A (compact) real form of $G$ is a $C^\ast$ Lie gp, subst HCG (compact) such that $\text{Lie} H^C \subset \text{Lie} H^R$ and $H^C_C = 0$.

Ex: $SL(2; C)$ has real forms $SL(2; R)$ and $SU(2)$ (not isomorphic).

Ex: $SL(2; R)$ not compact though.

Thm: If $G$ has a compact real form, then $\text{Rep}_C G$ is semisimple.

Pf: If $\pi_1(G) = 1$ then some rep is abelian. Every let $\pi \cong \pi_1 G$. Then $\pi$ is comp real form of $G$.

Rep of $G$ is compact. Now $\text{Rep}_C G \cong \text{Rep}_C \pi$ as trace map is fixed by $\pi$.