Exercises

1. Let $V, W, X$ be weight reps of $\mathfrak{sl}_2$ (possibly $\infty$-dimensional).

   Suppose the weights of $V$ and $W$ are disjoint (i.e. $VD_1 + 0 \Rightarrow WD_1 + 0$ and vice versa).

   Show that $\text{Hom}(V, W) = 0$. Show that any sequence $0 \rightarrow V \rightarrow X \rightarrow W \rightarrow 0$ splits.

   (This is different from showing $\text{Ext}^1(W, V) = 0$, b/c we assume that $X$ is weight.)

2. Suppose that $\lambda \neq \mu \in \mathbb{C}$ and either $\text{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0$ or $\text{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$. Prove that $\lambda \in \mathbb{Z}$ and $\mu = -\lambda - 2$.

   Hint: Use the Casimir operator, and exercise 1.

   By $\text{Ext}^1$ we mean extensions within the abelian category of weight reps of $\mathfrak{sl}_2$.

3. Draw the diagram for the weight rep $\Delta(-1) \otimes V_1$.

   Prove it is a nontrivial extension of $\Delta(0)$ and $\Delta(-2)$. (in which order?)

4. Draw the diagram for $\Delta(-1) \otimes V_2$. Show that $\Delta(-1) \otimes V_2 \cong \Delta(-2) \otimes X$.

   when $X$ is a nontrivial extension of $\Delta(1)$ and $\Delta(-3)$.

5. Let $H^*$ be a graded vector space, and $e^k: H^i \rightarrow H^{i+2}$ a degree-2 map (for each $i$) such that $e^k: H^{-k} \rightarrow H^k$ is an isomorphism. Prove that $\text{Ext}(e^k, e^l)$ of degree $0, -2$ respectively s.t. $\{e^k, e^l, e^m\}$ is an $\mathfrak{sl}_2$-triple.

   (Do NOT try to find a "magic formula" for $f$! Just be brutal and do what must be done.)

6. Let $H^*$ have basis $\{\lambda \mid \lambda \text{ is a partition fitting inside a } 3 \times 3 \text{ rectangle}\}$

   where $\lambda \in \mathbb{Z}$ is in degree $-g + 2k$.

   $H^0 \quad H^1 \quad H^2 \quad \ldots \quad H^i \quad \ldots$

   $C \phi \quad C \psi \quad C \psi^2 \quad \ldots \quad C \psi^i \quad \ldots$
Define $e^{i\theta} \to H^{ij,k}$ by $e^{i\lambda} = \sum (I+D)$, the sum of all ways to add a box to $\lambda$ and make a partition in a $3 \times 3$ rectangle.

a) Prove that $H^{ij,k}$ is an $8_2$ reps as in exercise 5.

b) Compute its character $+\mu$ multiplicity of irreducible in its decomposition.

7. a) If $\nu \leq \nu' \leq \lambda$ and $\nu \leq \lambda' \leq \lambda$, show that $\nu \leq \lambda$.

b) Show that $\text{ch}(\nu \otimes \lambda) = \text{ch}(\lambda) \text{ch}(\nu)$ in multiplication in $\mathbb{Z}[q, q^{-1}]$.

8. a) Let $[IJ] = \frac{q^{i-1} - q^{j-1}}{q^i - q^j} = q^{i-1} + q^{i-3} + \ldots + q^{i-2j} + q^{i-2j}$. $[IJ]_1 = 1$ $[0] = 0$.

Find a formula for $[IJK]_2$ in terms of quantum number.

b) Find a formula for $[IJK]$ which works for both $\lambda$ and $\lambda'$.

Your formula shall involve only $[IJ]_1$ for $d = 1$.

c) Decompose $V_0 \otimes V_0$ into irreducibles.

9. Check that $c = ef + fe + h^2/2$ is in the center of $U(gl_2)$.

10. Let $c \in \mathbb{Z}[A]$ for an algebra $A/\mathbb{C}$ and let $V, W$ be reps of $A$ st. $c$ has generalized eigenvalues $\lambda, \mu$ respectively.

If $\lambda = \mu$, prove that $\text{Hom}(V, W) = 0 = \text{Ext}^1(V, W)$.

11. Let $A = \mathbb{F}_q A_{1,2} \mathbb{F}_q A_2 \ldots \mathbb{F}_q A_{r-1} A = 0$ be a filtered algebra. Let $B = \bigoplus_{i=0}^{r-1} A_i$ be the associated graded.

Confirm that $B$ is a graded algebra, using the multi-definition in class.

Def: When $B$ is commutative, $A$ is called a short-commutative.
12. Suppose $A$ is almost commutative. Then if $x \in A$, $y \in A$, we know $xy - yx \in \text{End}_A$, $A = \text{End}_A$, or else $B$ would not be commutative. Define

$$B_i \odot B_j \rightarrow B_{ij}^{-1} \quad \exists x, y^2 = \overline{B}_1 - x_0$$

a) Check that $B_i, B_j$ is well-defined.

b) Check that $B_i, B_j$ is a Lie bracket.

c) When $A = U(3)$, check that $(B_i, B_j, B_k)$ is the Lie algebra $\mathfrak{u}(3)$.

d) Check that $B_i, B_j, B_k$ is a derivation on $B$ for any homogeneous $x \in B$.

e) Compute $B_i, B_j, B_k$ for $U(3)$.  

13. Let $R$ be the ring defined by generators and relations:

$$R = \mathbb{C}[x, y, z] / \begin{cases} xy = z^2 + 2yz + yx \\ y = zx \\ yz^2 = z^3(x + 2) \\ xz = z^2 + x \\ z = yz + z \\ \end{cases}$$

Use the Bergman diamond lemma to find a basis for $R$.  

(Use the relation \text{LHS} \rightarrow \text{RHS})

- What are the irreducible?
- What is the partial order?
- What are the ambiguities?
- Give a quick formula for $XZ^k$ in terms of irreducibles.
- Resolve all the ambiguities.

14. The Temperley-Lieb algebra $TL_n$ has presentation

$$TL_n = \langle v_i \rangle / \begin{cases} v_i^2 = 3v_i, \quad v_i v_{i+1} = v_{i+1} v_i, \quad v_i + v_{i+1} - v_i v_{i+1} - v_{i+1} v_i = 0, \quad v_i v_j = v_j v_i \quad \text{for } |i-j| \geq 2. \\ \end{cases}$$

Does the Bergman diamond lemma apply? Why or why not?
15. Check $\otimes$-Hom adjunction for Rep after any locally.


17. An element $cH$ is group-like if $\Delta(c) = c \otimes c$.
   is primitive if $\Delta(c) = c \otimes 1 + 1 \otimes c$.

   a) Prove that if $c$ is group-like then $E(c) = 1$ and $S(c) = c^{-1}$.

   b) Prove that the group-like elements form a group $H^{gp} C H$.

   c) Prove that if $X$ is primitive then $E(X) = 0$ and $S(X) = -X$.

   d) Prove that primitive elements form a lie algebra $H^{prm} C H^{lie}$.

18. A Hopf ideal $I$ is an ideal in $ICH$ such that:
   1. $\Delta(I) \subseteq I \otimes H + H \otimes I$
   2. $E(I) = 0$

   a) Show that (2) is redundant when $I$ is proper.

   b) Show that $H/I$ is a Hopf algebra.

   c) Show that the ideal generated by $(XY - YX - [XY])$ is inside $T(V)$

   for $X, Y \in V$ is a Hopf ideal.

   d) Show $cog$ is an ideal in a lie algebra. Prove that the ideal generated

   by $c$ in $U(c)$ is a Hopf ideal. What is the quotient?

19. a) Let $H = k[x]/x^2$ for any field $k$, with deg $d = 1$. Prove that this

   graded algebra is a Hopf algebra in the category of super vector spaces.

   b) Repeat for $H = \Lambda^* (k^n)$ for any $n$.

20. Well known: $H_1 = F_p [x]/x^p = 0$ $H_2 = F_p[y]/y^p = 1$ arc isom as algebras (try $y = x^p$).

   They are Hopf: $x$ primitive $y$ group-like.

   Are $H_1$ and $H_2$ isomorphic as Hopf algebras?