1. Let $G$ be a simply-connected, connected Lie group. Prove that

$$\text{Rep}_{\mathfrak{g},\text{smooth}}(G) \cong \text{Rep}_{\mathfrak{g},\text{man}}(\text{adj})$$

for $\text{adj} = \text{Lie}G$.

When $G$ is complex, prove that

$$\text{Rep}_{\mathfrak{g},\text{holo}}(G) \cong \text{Rep}_{\mathfrak{g},\text{holo}}(\text{adj})$$

2. Prove that the complexification $\text{adj}_C$ is a real Lie algebra.

3. Prove that $\mathfrak{sl}(n)_C \cong M(n;C)$

4. Prove that $\mathfrak{so}(n)_C \cong \{ X \in \mathfrak{gl}(n;C) | X + X^T = 0 \}$

   Where
   
   $$X = \begin{bmatrix}
   & & & & \\
   & \ddots & & & \\
   & & \ddots & & \\
   & & & \ddots & \\
   & & & & \\
   \end{bmatrix}
   $$

   (This is like the transpose, except flipped instead of \(\gamma\))

   (Hint: Use last week's HW to show $\mathfrak{so}(n)_C$ is isomorphic to $\mathfrak{so}(B)$ for some $B$.)

5. Prove that $\mathfrak{su}(2) \neq \mathfrak{sl}(2;\mathbb{R})$. More concretely, find

   $$X \in \mathfrak{sl}(2;\mathbb{R})$$

   such that $[X,Y] = X\gamma$ for some $X \in \mathbb{R}$.

   Prove that this does not happen in $\mathfrak{su}(2)$.

6. Find an example of a Lie group $G$ where $\text{Rep}G \neq \text{Rep} \text{adj}$

   when the representations are allowed to be infinite dimensional.

   (Hint: When $\text{adj}$ is abelian, what does $\text{Rep} \text{adj}$ look like?)

   What is wrong with the expected map?
7. Prove that $\text{Der}A$ is an ideal inside $\text{Der}A$ for a Lie algebra $A$. (Recall: $\text{Der}A$ is the image of $ad \to \text{Der}A$.)

8. Classify all 3D Lie algebras over any field $F$ for which $[\gamma, \delta] \neq 0$.

9. a) Prove that if $H \subseteq G$ then $\text{Lie H} \subseteq \text{Lie G}$ is a subalgebra.
      b) What is $\text{Lie H} \subseteq \text{Lie G}$ on ideal? 

10. Let $X \in \mathfrak{gl}(n;F)$ have $n$ distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that the eigenvalues of $ad_X \in \mathfrak{gl}(n;F)$ are the $n^2$ scalars $\lambda_i - \lambda_j$ (possibly with multiplicity when $\lambda_i = \lambda_j$).

11. Prove that $[e_{1n}, e_{jn}] = s e_{ij}$ and $[e_{1n}, s e_{ij}] = s e_{ij}$, over $\mathbb{R}$ or $\mathbb{C}$.

12. Show that $[X, X] = 0 \forall X \iff [X, Y] = -[Y, X] \forall X, Y$ outside of characteristic 2.

13. Let $B = \begin{pmatrix} \lambda & \ast \\ \ast & \ast \end{pmatrix}$ in $\mathfrak{gl}_n$.

   Compute $B = \text{Lie}B$ and $U = \text{Lie}U$.