Thm 1: \( G, H \) are matrix Lie groups, \( g = \text{Lie} G, h = \text{Lie} H \). (Mathify font)

Given \( \phi: G \rightarrow H \) have \( \phi(g) = h \) a linear map. Then

1. \( \phi(e^x) = e^{\phi(x)} \) \quad \forall x \in G \)
2. \( \phi(A^T x^{-1}) = \phi(A)^{-1} \phi(x)^{-1} \) \quad \forall x \in G \)
3. \( \phi( [X,Y] ) = [\phi(X), \phi(Y)] \) \quad \forall x \in G \)
4. \( \phi(x) = \frac{d}{dt} \bigg|_{t=0} \phi(e^{tx}) \)

And moreover, \( \phi \) satisfies the chain rule, i.e., if \( G \xrightarrow{\phi} H \xrightarrow{\psi} K \) then \( \psi \circ \phi \).

Pf: 4. This is the chain rule. \( R \xrightarrow{e^x} G \xrightarrow{\phi} H \) yields \( \frac{d}{dt} \bigg|_{t=0} \phi(e^{tx}) \).

1. \( \phi(e^x) \) is a 1-parameter family so by thm of before, it is \( t \rightarrow e^{tz} \) for some \( z \in L \). But then \( z = \frac{d}{dt} \bigg|_{t=0} \phi(e^x) \).

Aside: If we didn't know about derivatives of smooth maps, we could use 1 to deduce 4.

Then we need to check that it is a linear map (chain rule is fairly clean)

rescaling is easy. \( \phi(A^T y) \) is \( z \) s.t.: \( z = \phi(e^{xy}) = \phi(I(n,\mathbb{x},\mathbb{y})) \)

\( e^{\delta e^{xy} + y} = \lim_{\delta \rightarrow 0} \left( \phi(e^{\delta e^{xy}}) \right) = \lim_{\delta \rightarrow 0} \left( \phi(e^{\delta x}) \phi(y) \right) \)

2. \( \phi(e^{A^T x}) = \phi(Ae^x) = \phi(I(n,\mathbb{x},\mathbb{y})) = \phi(I(n,\mathbb{x},\mathbb{y})) \phi(y) = e^{xy} \phi(y) \)

3. \( [X,Y] = \frac{d}{dt} \bigg|_{t=0} e^{xy} e^{tx} \quad \forall x \in G \)

\( \frac{d}{dt} \bigg|_{t=0} e^{xy} e^{tx} \) is a linear map of previous derivates:

\( d\phi = \frac{d}{dt} \bigg|_{t=0} \phi(e^{tX} \phi(y)) e^{-tX} \) s.t.:\( d\phi = \frac{d}{dt} \bigg|_{t=0} \phi(e^{tX} \phi(y)) e^{-tX} = [d\phi(X), d\phi(Y)] \)
So get a map of $y$ with many nice properties. But it turns out the only really

important one is (3) that it preserves the bracket.

**Theorem 2.1** If $y_1, y_2 : G \to H$ and $\delta y_1 = \delta y_2$ then $y_1 = y_2$.

*Proof:* If the U of $I \in G$ st. every $g \in U$ is $e^x$ for some $x \in \mathbb{R}$.

**Lemma:** If $G$ is connected and $U$ is a nbhd of $I$ then every $g \in G$ is in $U^k$ for some $k$, $U^k = \{g \cdot g_2 \mid g \cdot g_2 \in U\}$.

**Proof:** Check a path $I \to g \cdot g_2$, use connectedness + connectedness of $G \
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So every $g \in G$ has the form $e^{x_1} e^{x_2} \cdots e^{x_k}$ for some $x_i \in \mathbb{R}$.

Then $y_1(g) = y_1(e^{x_1} e^{x_2} \cdots e^{x_k}) = (y_1(e^{x_1}), y_1(e^{x_2}), \ldots, y_1(e^{x_k})) = e^{\delta y_1(x_1)} e^{\delta y_1(x_2)} \cdots e^{\delta y_1(x_k)}$.

Similarly $y_2(g) = e^{\delta y_2(x_1)} e^{\delta y_2(x_2)} \cdots e^{\delta y_2(x_k)}$.

$\square$

**Theorem 3.1** Suppose one has a map $f : G \to H$ st. $f(I \cdot g) = \delta f(g)$.

*Proof:* If $G$ is connected, simply connected, then $f$ is a homomorphism with $\delta f = 0$.

Let $U$ be a nbhd of $I$ where $\exp$ is diff. Define $\psi : U \to H$ by

$\psi(u) = e^{-f(b)u}$.

so $\delta (\psi)_U = 0$. Let $V$ be a nbhd of $0$ in $U$ matching $U$.

**Claim 1:** If $u, v \in U \cap V$ then $(f(u) + f(v)) = f(u + v)$.

*Proof* (BCH formula): $u = e^x \cdot e^y = e^x e^y = e^{x + y} = e^{x + y + \frac{1}{2} [x, y] + \cdots}$.

Then $f(u + v) = e^{-f(b)(u + v)} = e^{-f(b)u - f(b)v}$.

$\square$
Now let \( p: [0,1] \rightarrow G \) be a path, and choose \( t_0 = 0, t_1, \ldots , t_k = 1 \) s.t. \( p(t)p(t_{k-1})^{-1} \in U \) (cancellation against \( \text{corners} \) by \( \text{path} \)).

For more precisely, \( p(s)p(t_{k-1})^{-1} \in U \bigwedge s \in [t_{k-1}, t_k] \) (Don't keep \( U \) out of sight).

Now define \( \Phi(p) = \Phi(p(t_k)p(t_{k-1})^{-1}) \Phi(p(t_{k-1})p(t_{k-2})^{-1}) \cdots \Phi(p(t_1)) \) \( \hat{=} \hat{ } \).

**Claim 2:** The depends on the path \( p \), but not on the choice of times \( t_i \).

**PF:** Gap to show that inserting a new time doesn't change the answer.

Then you can mutually refine two choices.

But \( \Phi(p(t_{k-1})p(t_1)^{-1}) = \Phi(p(t_{k-1})p(t_{k-2})^{-1}) \Phi(p(t_{k-2})p(t_1)^{-1}) \) by local homomorphism.

**Claim 3:** Homotopic paths have \( \Phi(p) = \Phi(q) \).

**PF:** One can deform paths smoothly by adding up little squares with \( U \) translates.

Then \( \Phi(p(y_1 x^{-1})) = \Phi(y_1 z_1^{-1}) \Phi(z_1 x^{-1}) = \Phi(y_1 z_1^{-1}) \Phi(z_1 x^{-1}) \) by local hom, so that nearby paths agree.

Compos of \( D^2 \) yields a finite number of small changes to get from \( p \) to \( q \).

**Part of proof:** So define \( \Phi(p) = \Phi(p) \) for any path to \( g \) (relevant which, since \( \Phi \) is simply connected).

Company \( \text{comp} \) with \( \text{path} \) get a path to \( g \).

Using a time decomposition including \( g \), we see that

\[ \Phi(g h) = \Phi(g h) \Phi(h^{-1}) \Phi(h) = \Phi(g) \Phi(h) \] so it is a homomorphism.

Then smoothness follows from smoothness near \( I \), since \( \Phi \) near \( g \) is

\[ \Phi(g) \cdot \Phi(\text{near } I) \] and left multiplies.