McKay Correspondence: An interesting bijection b/w finite subgroups \( \text{SU}(2) \) of \( \text{SU}(2) \) and affine Dynkin diagrams = "McKay graphs" for short.

Quickly:
\[
\text{SU}(2) = \left\{ \begin{array}{c} (a, b) \in \text{Mat}_{2 \times 2}(\mathbb{C}) \\ |a|^2 + |b|^2 = 1 \end{array} \right\}
\]

McKay graphs: The subscript is the \( (-1) \) of vertices, why later. They are very special for many reasons!

\[\widehat{A}_n\]  
\[\widehat{D}_n\]  
\[\widehat{E}_6\]  
\[\widehat{E}_7\]  
\[\widehat{E}_8\]

Special case: \[\widehat{A}_n\]  
\[\text{if } n \neq 4\]

Simply laced = no loops or multiple edges.

The story will be: given \( G \subseteq \text{SU}(2) \), consider \( \text{Rep}_G \) and the special structure it inherits.

Then extract combinatorics to get a graph \( G_6 \) and prove it is a McKay graph.

**Idea**: Classify things by classifying possible categories of representations (w/ implicit structure).

One can also classify subgroups of \( \text{SU}(2) \) directly, and we'll do that first for context. The interesting part of the theorem is the bijection, not the classification.

We'll review \( \text{SU}(n) \) etc. soon, but more exciting, start w/ a related problem.

### 8.2. Finite subgroups of \( \text{SO}(3) \)

\( \text{SO}(3) = \) group of transformations of \( S^2 \) = things you can do to a globe.

(Again, more review soon).

Space \( H \subseteq \text{SO}(3) \) finite, and consider orbit under \( H \) of a point in \( S^2 \). This must be a regular polyhedron (all points are the same).

**Thm** (Theaetetus, *Elements*, see Euclid's Elements): The only 3D regular polyhedron are:

- \( d_4 \)
- \( d_6 \)
- \( d_8 \)
- \( d_{12} \)
- \( d_{20} \)

- Tetrahedron
- Cube
- Octahedron
- Dodeca-
- Icos-

**Pf**: It's slick! See wikipedia for rough outline. (Ask: Why Platonic solids? Also see wiki)

\( \uparrow \text{nice but hard.} \)
These solids have finite symmetry groups inside $O(3)$ and $SO(3)$

**Tetra:**

- Size: $S_4 = (-A_3)$, 24
- Reflections through edge 1-2 induce permutation on vertices (34).
- Get all permutations of vertices.
- $\det(\text{ref}) = -1$ so $\det = \text{sgn}$

- $O(3)$ and $SO(3)$ also include reflections which turn $S^2$ inside-out.

**Cube:**

- Size: $S_{3} = (B_3)$, 48
- Signed symmetric group
- $S_n$ = permutations of $\{\pm 1, \pm 2, \ldots, \pm n\}$
- $i \rightarrow j \Rightarrow -i \rightarrow -j$
- (i.e. they are "linear")
- $|S_n| = |S_n| |\{\pm 1, \ldots, \pm n\}| = n! 2^n$
- Think of $\{\pm 1, \pm 2, \ldots\}$ as the 6 faces.

**Octahedron:**

- Same as cube!
- Cube + Octa are dual polyhedra.
- Vertices ↔ faces, edges ↔ edges. Dual polyhedra have same symmetry group.
- Tetra is dual to tetra. Dodeca is dual to Icosa.

**Dodeca/Icosa:**

- Size: $H_3 = 120$
- $A_5 = (I)$, 60

(A_3, B_3, H_3 are the rank 3 irreducible finite Chevalley groups. Later in course, $A_n$ groups generated by "reflections".)

Any other regular polyhedra in $S^2$? Sure... 2D (and 1D) polyhedra.

- Get $C_n = \mathbb{Z}/n\mathbb{Z}$ and also $D_n$, size 2n, in both $O(3)$ and $SO(3)$ (Flipping in 2D can come from rotation in 3D).

**Prop:** The finite subgroups of $SO(3)$ are: $C_n$, $n \geq 1$, $D_n$, $n \geq 2$, $T$, $O$, $I$

- Size: $n$ or $2n$, 12, 24, 60

Like McKay graphs: two infinite families and three sporadic groups.
33 Linear algebra review: the groups \( \text{GL}(n) \), \( \text{O}(n) \), \( \text{U}(n) \), \( \text{SO}(n) \)

**Def:** Let \( \mathbb{F} \) be a field and \( V \) a \( \mathbb{F} \)-v.s./f.f. \( \text{End}(V) = \{ \text{linear fns. } V \to V \} \)

\( \text{GL}(V) = \text{End}(V) \) is the subgroup of isomorphisms.

When \( V \equiv \mathbb{F}^n \) (i.e., a basis \( \{ e_1, \ldots, e_n \} \) of \( V \) is chosen), then one can identify

\[ \text{End}(V) = \text{Mat}(n, n; \mathbb{F}) \quad \text{GL}(V) = \text{GL}(n; \mathbb{F}) = \{ \text{invertible } n \times n \text{ matrices} \} \]

**Remind:** If \( A \in \text{Mat}(n, n; \mathbb{F}) \) then

\[
A = \begin{pmatrix}
A_{e_i e_j} \\
\vdots \\
A_{e_i e_n}
\end{pmatrix}
\]

Now, let \( \langle - , - \rangle \) be the standard symmetric bilinear form on \( A \). \( \langle e_i, e_j \rangle = \delta_{ij} \)

**Note:** \( A_{e_i e_j} = (A_{e_j e_i})^T \).

**It's a fact:** Matrix coefficients are detected by applying a bilinear form to test vectors.

**Def:** \( A^T \) is the adjoint transpose matrix, \( (A^T)_{ij} = A_{ji} \)

**Prop:** \( \langle Av, w \rangle = \langle v, A^T w \rangle \) — this is what makes it the adjoint w.r.t \( \langle -, - \rangle \)

**Def/Prop:** \( A \) is orthogonal if (TFAC)

1. \( A \) is invertible and \( A^{-1} = A^T \)

2. \( A \) preserves the form, i.e., \( \langle Av, Aw \rangle = \langle v, w \rangle \)

**PF:** \( \langle v, w \rangle = \langle v, (A^T A) w \rangle \).

By plugging in test vectors \( v = e_i, w = e_j \) get

\[
\begin{align*}
\langle v, w \rangle &= \langle e_i, e_j \rangle = \delta_{ij} \\
&\iff (A^T A)_{ij} = \delta_{ij} \\
&\iff A^{-1} = A^T.
\end{align*}
\]

**Def:** \( \text{O}(n; \mathbb{F}) = \{ A \in \text{GL}(n; \mathbb{F}) \mid A \text{ is orthogonal} \} \)

\( \text{SL}(n; \mathbb{F}) = \{ \text{ } \mid \det A = 1 \} \)

\( \text{SO}(n; \mathbb{F}) = \text{O} \cap \text{SL} \)

**Why do we like \( \text{O}(n) \)?** The standard form on \( \mathbb{F}^n \) is positive definite, i.e.

\( \langle v, v \rangle \geq 0 \) with \( \langle v, v \rangle = 0 \implies v = 0 \).

Consequently, we have a notion of lengths and angles, and \( \text{O}(n; \mathbb{F}) \) preserves lengths + angles.

**Ex:** \( \text{O}(3) \) contains \( \text{SO}(3) \) (generated by reflections \( \sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) rotations \( \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)).

\( \text{SO}(3) \) only has rotations. (Every rigid symmetry of a globe is a rotation.)

Not true for \( \text{SO}(n), n > 3 \).
But $O(n;\mathbb{F})$ isn't so great. Pos def? What does $(v, v) > 0$ even mean?

In $O(n; \mathbb{C})$, if $(v, w) \in \mathbb{R}_{\geq 0}$ then $(v, iv) = i^2(v, w) \in \mathbb{R}_{\geq 0}$.

When working over $\mathbb{C}$ can either have pos def or symmetric bilinear, but not both.

Def: A pairing on $\mathbb{C}^n$ is **sesquilinear** if $\forall v, w, \in \mathbb{C}^n$ $(Av, w) = \overline{A^*(v)}(w)$ anti-linear

(mean 1.5-linear) instead of bilinear

It is **hermitian** if $(v, w) = \overline{(w, v)}$ (replace symmetric)

(could also replace symmetric $v$) Skew-hermitian, $(v, w) = -(w, v)$

Now if $z \in \mathbb{C} \subseteq \mathbb{C}^*$ then $(zv, zw) = \overline{\overline{z}z}(y, w) = (y, w)$.

Let $\mathbb{C}^n$ have the standard sesq. herm. form $(e_i, e_j) = \delta_{ij}$. Then $(y)v \in \mathbb{R}_{\geq 0}$ with

$(y)v = 0 \iff v = 0$.

Now, w.r.t. $(-, -)$, the adjoint matrix $A^*$ is $A^*$.

$\begin{align*}
(Av, w) = (v, A^*w)
\end{align*}$

Def/Prop: $A$ is **unitary** if (TFAE)

1. $A$ is invertible and $A^{-1} = A^*$
2. $(v, w) = (Av, Aw) \forall v, w$.

These form a group $U(n)$ the field $\mathbb{C}$ is implicit. Columns of $A$ are orthonormal.

$U(n) \triangleq SL(n; \mathbb{C}) = SU(n)$.

**Note:** Just bc $U(n) \subseteq \text{Mat}_{n \times n}(\mathbb{C})$ does NOT mean it is "complex-linear."

If $A \in U(n)$ and $z \in \mathbb{C}^*$ then $zA \in U(n)$ in general $U(n)$ is actually a real manifold, but NOT a complex manifold. Not even even-dimensional.

§4: $SU(2)$ and $SO(3)$

Fact: Every $A \in SU(n)$ is diagonalizable. (or just $U(n)$)

Consequence: $\mathbb{Z}(SU(n)) \subseteq \text{diag}$ and matrices. But the only central diagonal matrices are scalars $zI$ for $z \in \mathbb{C}^*$. Then unitary $\implies z \in \mathbb{S} \subseteq \mathbb{C}^*$.

$A = zI$ so $z \in \mathbb{Z}/\mathbb{Z} \subseteq \mathbb{S}$.

$\mathbb{Z}(SU(n)) \cong \mathbb{Z}/\mathbb{Z}$
Now, \( \text{SU}(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\} \)

\[
\begin{cases}
\left( \frac{a}{\sqrt{d}}, \frac{c}{\sqrt{d}} \right) \quad \text{if} \quad d > 0 \\
\left( -\frac{c}{\sqrt{-d}}, -\frac{a}{\sqrt{-d}} \right) \quad \text{if} \quad d < 0
\end{cases}
\]

Topologically, this is \( S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2 \).

Now \( \mathbb{Z}(\text{SU}(2)) = \left\{ \pm \mathbb{I} \right\} \cong \mathbb{Z}/2\mathbb{Z} \). Moreover, \(-I\) is the \underline{only} involution in \( \text{SU}(2) \).

If \(-I\) is only diagonal involution, but central. \( AI = IA \Rightarrow A = -I \).

Prop. There is an s.e.s.

\[
0 \to \mathbb{Z}/2\mathbb{Z} \to \text{SU}(2) \to \text{SO}(3) \to 0
\]

Quick version, more in exercises, and generalization later in the class:

\[
V = \text{traceless Hermitian matrices} = \text{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}
\]

Let \( A^* = A \)

Define a pairing on \( V \) via \( (X, Y) = \text{Tr}(XY) \). \( \underline{Claim:} \) This is non-degenerate!

Claim: Given \( A \in \text{SU}(2), X \in V \) then \( AXA^{-1} \in V \) so \( \text{SU}(2) \subset V \). Moreover, \( (AX, AY) = (X, Y) \) so \( \text{SU}(2) \to O(V) \cong O(3) \)

Can confirm:
1. \( U \) has image \( \text{SO}(3) \)
2. \( \ker \phi = \mathbb{Z}(\text{SU}(2)) \).

\[5\] Finite subgroups of \( \text{SU}(2) \):

\[
\mathbb{G} \subset \text{SU}(2) \quad \overset{\phi}{\longrightarrow} \quad H = \phi(G) \subset \text{SO}(3)
\]

Either
1. \( \mathbb{Z}/2\mathbb{Z} \subset G \) so \( |G| = 2|H| \), \( G = \phi^2(H) \)
2. \( \mathbb{Z}/2\mathbb{Z} \not\subset G \) so \( |G| = |H| \), \( \phi \) is an isomorphism.

Note that \( 1 \Leftrightarrow |G| \) is even, since \(-I\) is unique involution. \( \text{Call} \, 1 \Leftrightarrow \, \text{involutory} \).

Note that \( |H| \) is even \( \Rightarrow |G| \) is even.

We have classified \( H \subset \text{SO}(3) \), and \( |H| \) is even unless \( H = G \) for \( n \) odd.

\( \text{SU}(2) \) does contain cyclic groups, and it is easy to see that they are all conjugate of \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) in the diagonal matrices.
Prop: The finite subgroups of SU(2) are (isomorphic, by conjugation, to):

\( C_n \) cyclic \( \{ (\cos \frac{2\pi k}{n}, \sin \frac{2\pi k}{n}) \}_{k=0}^{n-1} \)

\( = \mathbb{Z}/n \mathbb{Z} \) when \( n \) even.

\( 4n = D_{2n} \) binary dihedral \( \{ (\cos \frac{2\pi k}{n}, \sin \frac{2\pi k}{n}), (\cos \frac{2\pi k}{2}, \sin \frac{2\pi k}{2}) \}_{k=0}^{n-1} \)

\( g^4 = h^2 = 1 \) (\( g^2 = -1 \) is central, involution)

Notes: \( T^* \) same size as tetrahedron symmetry group in \( O(3) \), but NOT the same,

\( e.g. \ T^* \neq S_4 \), \( S_4 \) has many involutions, \( T^* \) has one.

Cool Fact: These groups all have presentations which lift to presentations of double cover:

\[ C_n = \langle a | a^n = 1 \rangle \]
\[ D_{2n} = \langle a, b | a^n = b^2 = (ab)^n = 1 \rangle \]
\[ T = \langle a, b | a^2 = b^3 = (ab)^3 = 1 \rangle \]
\[ O = \langle a, b | a^2 = b^3 = (ab)^4 = 1 \rangle \]
\[ I = \langle a, b | a^2 = b^3 = (ab)^5 = 1 \rangle \]

Ex: \([\begin{array}{cccc} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{array}]\)

\( a = (34)(12), b = (134) \)

Qn: When can you binaryize a group presentation (and get a double cover, not something trivial)?

Ex: Binaryizing \( C_2 \times C_4 \) gives the generalized quaternion group \( \mathbb{Q}_{2^k} \).

\( Q_8 \) is binary \( C_2 \times C_2 \).
McKay Correspondence

"Idea": Identify structure, transform into combinatorics!

Given \( G \subset SU(2) \), have \( G \subset C^2 \subset V \). A very nice resp.

Def: Let \( \Gamma_G \) be the labeled graph defined as follows:

\[ \begin{align*}
& \bullet \text{ vertices } \leftrightarrow \text{ rep } \sigma \text{ of } G \text{ /iso} \\
& \bullet \text{ label } d_i \in \mathbb{Z}_{\geq 0} \text{ on a vertex } i = \dim V_i \\
& \bullet \text{ if } \sigma_i \text{ appears in } V_c \text{ w/multiplicity } m_i, \text{ "branching graph" } M_{i \to j} \\
& \text{ what happens to reps after applying a functor.}
\end{align*} \]

We will soon prove: \( M_{i \to j} = M_{j \to i} \), so may as well consider an undirected graph.

\[ M_{i \to i} = 0 \] unless \( G \) is trivial group, where \( M_{\text{trivial}} = 2 \). No loops.

Def: Let \( \Gamma \) be an undirected graph whose vertices are labeled by positive integers. We call \( \Gamma \)

an Mckay graph if it satisfies:

1) basepoint: \( \exists \) distinguished vertex \( 0 \), \( d_0 = 1 \).
2) harmonic: \( 2d_i = \sum_j d_j M_{i \to j} \), \( M_{ij} \neq 0 \) if edge \( i \to j \) (possibly 0)
3) connected
4) no loops

Thm: a) If \( G \subset SU(2) \) is nontrivial, then \( \Gamma_G \) is a Mckay graph.

b) The Mckay graphs are \( \tilde{A}_n, \tilde{S}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \).

c) \( G \to \Gamma_G \) is a bijection \( \{ \text{ nontrivial } G \subset SU(2) \} \leftrightarrow \{ \text{ finite Mckay graphs} \} \)

The outline is:

1. Use rep-theory to prove a.
2. Use combinatorics to prove b. This is on the exercise.
3. Just match it up to prove c. Matching is easy; \#Irreps = \#conjug classes.

This proof of c is unsatisfactory! One would rather provide an interesting construction which
takes a Mckay graph and magically ("actually") produces a subgp of \( SU(2) \).

Later in the course we will have nice bijections with such magic constructions! 

Remark: With the exception of \( \tilde{A}_1 = \tilde{E}_8 \), every Mckay graph is simply laced.
Ex: \( G = C_n = \{(\alpha^0, \alpha^1) \mid \alpha^0, \alpha^1 \in \mathbb{Z}_n \} \) where \( \alpha = (\alpha^0, \alpha^1) \). 

\( G \) is abelian \( \Rightarrow \) all irreps are 1D. \( V_k = C^k \), \( \alpha x = \alpha^k x \).

Then \( V_k \otimes V \) has basis \( \{x \otimes e_1, x \otimes e_2\} \) where \( a(x \otimes e_1) = a(x) \otimes e_1 = S_x \otimes S_{e_1} = S_{x \otimes e_1} \) and \( a(x \otimes e_2) = S_x \otimes S_{e_2} = S_{x \otimes e_2} \).

So \( V_k \otimes V = V_k \oplus V_{k-1} \).

\[ \Rightarrow \quad V^2 = V_3 \oplus V_2 \oplus V_1 \]

\[ \Rightarrow \quad \text{spec. case: } n = 2 \]

**Rep Theory**

**Base case:** Any group \( G \) has a trivial rep \( V_0 = C^1 \) where \( g \cdot 1 = 1 \forall g \in G \).

**Harmonic:** \( \dim V_c \otimes V = 2 \dim V_c \).

By semisimplicity, \( V_0 \otimes V = \bigoplus V_j \) for some multiplicities \( \# M_v = j \), and so \( 2 \dim_i = \dim (V_i \otimes V) = \sum j M_v \rightarrow g \).

**Interesting fact:** \( M_v \rightarrow g = M_{g^{-1} v} \). Let's prove it.

**Prop:** \( V \) is self-dual, i.e., \( V \cong V^* \) as \( G \)-rep.

**Rik:** This is really what the unitary group gives you!!

**Pf:** \( V \) has self-hermitian form \((\cdot, \cdot)\).

Use it to identify \( V^* \cong V \) as \( V \)-vis.

\( (v, \cdot) = f_v \leftrightarrow v \)

\( f_i \rightarrow v_i \)

**Action of \( A \in GL(V) \) on \( V^* \) is**

\[ A^* f(v) = f(A^{-1} v) = (A^{-1} v, A^{-1} v') = \left( A^{-1} f v, v' \right) \]

\[ f_{A^{-1} v} (w) \]

So \( V^* \) is, as a \( GL(V) \) rep, isomorphic to \( V \) with \( A \cdot v = (A^{-1})^* v \).

When \( A \in GL(V), \quad (A^{-1})^* = A \) and get usual action.

**Side Notes** (Basic Concepts)

**Defs:** Subrep, Semisimple.

**Irreducible repn. Semisimple.**

**Tensor product.**

**Dual repn. Character. Inner product of characters**

**Pf:** \( \chi_{V^*} = \chi_V^* \) for any rep \( V \).

\( V \cong V^* \iff V = V^* \) so enough to show that \( \chi_V (g) \in R \quad \forall g \in SU(2) \).

But \( g \sim (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) \Rightarrow \chi_V (g) = \frac{3 + i}{2} \in R \) as desired.
Two more ingredients:

**Schur's Lemma** (holds over C): \( \dim \text{Hom}(V_i, V_j) = \delta_{ij} \)

\[ \Rightarrow \text{if } W = \bigoplus V_i^m \text{ then } \dim W = \dim V_i = \dim \text{Hom}(V_i, W) = \dim \text{Hom}(W, V_i) \]

**Tensor-Hom adjunction:**

\[ \text{Hom}(W \otimes X, Y) \cong \text{Hom}_G(W, \text{Hom}(X, Y)) \]

**Consequence:**

\[ \dim \text{Hom}(V_j, V_i \otimes V) = \dim \text{Hom}(V_j \otimes V^*, V_i) = m_{ij} \text{ for } \bar{V} \neq V^* \]

---

Now loops: My proof of this sucks - I'll return to it. Not hard.

**Connected:** The hard part. Use a tricky character proof!

**Lemma:** \( V \) is irreducible \( \iff G \) is not abelian \( (\iff G \neq C_n) \)

**PF:** If \( V \) is reducible, \( V = C \times C \) as \( G \)-reps. Up to conjugation, \( x = e_1, y = e_2 \). \( G \) diagonal matrices, \( \cong \mathbb{S}^1 \) abelian.

Conversely, any abelian subgroup of \( SU(2) \) is conjugate to a diagonal subgroup. \( \square \)

We've already computed that \( G \neq C_n \) is connected, so let's assume \( V \) irreducible.

Also, this rules out \( C_n \), so we can assume \(-I \in G\).

**Connected** \( \iff \) Each \( V_i \in V \otimes k \) for some \( k \) \( \iff (X_{V_i}, X_{V \otimes k}) \neq 0 \) for some \( k \)

Now \( (X_{V_i}, X_{V \otimes k}) = \frac{1}{16} \sum_{g \in G} X_{V_i}(g) \overline{X_{V \otimes k}(g)} \)

We've seen \( X_{V_i}(e^{0 \cdot 0}) = 2 \Re S \), so \( X_{V_i}(g) = \begin{cases} 2 \quad g = I \\ -2 \quad g = -I \\ (e^{\pi i}, e^{2 \pi i}) \quad \text{else} \end{cases} \)

Now \(-I \in \mathbb{Z}(G) \) so \(-I \in \text{End}_G(V_i) = C \cdot \text{id}_{V_i} \), either \(-I \) acts by \( + \text{id}_V \) or \(- \text{id}_V \).

\[ X_{V_i}(-I) = \pm \dim V_i \quad X_{V_i}(I) = \pm \dim V_i \]

\[ \Rightarrow (X_{V_i}, X_{V \otimes k}) = \frac{1}{16} \left( \dim V_i \cdot 2^k + (\pm 1) \dim V_i \cdot (-2)^k + \sum_{g \neq I} X_{V_i}(g) \overline{X_{V_i}(g)} \right) \]
Restrict to $k$ even/odd st. $X_k(-2)^k$ is positive.

Then \[ (X_k, X_{k^2}) = \frac{1}{|G|} \left( 2 \dim V_i + \sum_{g \neq 1} X_k(g)(\frac{k}{2}) \right) \]

\[ \lim_{k \to \infty} = \frac{2 \dim V_i}{|G|} \neq 0 \text{ so some } (X_k, X_{k^2}) \neq 0. \]

The action of $-I$ is even, $-I \in G$. As noted, $-I$ acts on $V_i$ by either $+1$ or $-1$.

$-I$ acts on $V$ by $-1$, so acts on $V_i \otimes V$ (and any summand thereof) by the opposite sign.

$\Rightarrow \Gamma^*_G$ is bipartite! $-I$ acts on triv by $+1$.

$C_n$, $n$ even
$D_{n^2}$, $n$ even
$D_{n^2}$, $n$ odd

Now $-I$ acts by $+1$ $\Leftrightarrow$ action of $G$ factors through $H = \{e \in G \}$, so the black vertices give the graphs of subgroups of $SO(3)$.

$C_{n/2}$ $D_{n^2}$, $n$ even $D_{n^2}$, $n$ odd $T=A_4$ $O=S_4$ $I=A_5$

Rmk: Why no loops? No loops in a bipartite graph, so if $\Gamma^*_G$ has a loop then $|G|$ is odd.

But we know this means $G = C_n$ for $n$ odd, and we know $\Gamma^*_C$ has no loops unless $n=1$.

However, this is a ugly reason, relying on classification of subgps. I don't know a better reason.

$\text{Aut}(\Gamma^*_G) \Rightarrow$ Let $V_i$ be a curve of dim 1. Then $\otimes V^*$ is an invertible functor with inverse $\otimes V^*$. This functor preserves irreducibles, so it induces an automorphism of $\Gamma^*_G$. These automorphisms form a subgroup $\text{Aut}(\Gamma^*_G)$ which acts simply transitively on the vertices labeled 1.

Rmk! $\text{Aut}(\Gamma^*_G)$ except for $\hat{E}_7$ and $\hat{E}_8$.
McKay graphs are simply laced. Prop: Unless \( \Gamma = A_1 \), \( M_{i-j} = 0 \) or 1.

\[
2d_j = \text{md}_i + \sum d_{k} M_{j-k}
\]

\[
2d_i = \text{md}_i + \sum d_{k} M_{i-k} = \sum d_{k} (M_{i-k} + M_{j-k})
\]

\[
= 0 \iff m = 2 \quad = 0 \iff m = 2
\]

\( \Rightarrow \) both sides = 0, so \( m = 2 \) and graph is \( A_1 \).

Infinite McKay graphs One of our first tasks in Lie gp theory will be to prove that Rep\( G \) is semisimple when \( G \) is a compact Lie gp. This includes all finite groups.

The same rep theory results prove that \( \Gamma^* \) is a (non-finite) McKay graph.

Thm (Extended McKay Con.) \(
\{ \text{(Compact) subps of } SU(2) \} \iff \{ \text{McKay graphs} \}
\)

\[
\begin{array}{c}
\text{SU}(2) \iff E_6 \\
D_\infty = \langle (0,0), (1,1), (1,0) \rangle \iff D_\infty \\
\end{array}
\]
A classification of graphs

I. \( \Gamma \) is a proper subgraph of a McKay graph.

(\( \Rightarrow \) \( \Gamma \) is simply laced)
These are called simply laced Dynkin diagrams.

They are: (now subscript = # of vertices)

- \( A_n \)
- \( D_n \)
- \( E_6 \)
- \( E_7 \)
- \( E_8 \)

II. \( \Gamma \) is a McKay graph.

(No two contain each other)
Also called (simply-laced)
affine Dynkin diagrams
(w/ \( \tilde{A}_1 \))

III. \( \Gamma \) properly contains a McKay graph.

(Con remain vertices and/or edges to get a McKay graph.)
This is everything else.

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PF: Straight-up easy case-by-case analysis.
- Does it have multiple edges? (\( \tilde{A}_1 \))
- Does it have cycles? (\( A_n \))
- Does it fork more than once? (\( D_n \))
- If no forks, \( A_n \).
- Does it fork have more than 3 outputs? (\( D_4 \))
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Finally, the interesting part. Spouse

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3 cases:
\[
\frac{1}{p+\frac{1}{q}+\frac{1}{r}} = 1
\]
(\(2,2,n\)) is \( D_n \)
(\(2,3,3\)) \( E_6 \)
(\(2,3,4\)) \( E_7 \)
(\(2,3,5\)) \( E_8 \)
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\[
\frac{1}{p+\frac{1}{q}+\frac{1}{r}} = 1
\]
(\(2,3,6\)) \( E_6 \)
(\(2,4,4\)) \( E_7 \)
(\(3,3,3\)) \( E_8 \)
```

\[
\frac{1}{p+\frac{1}{q}+\frac{1}{r}} < 1
\]
Everything else.

Why is this classification important?
Def: Let $V_f$ be the $\mathbb{R}$-v.s. spanned by $x_i j$ in $\text{Vertices}(\Gamma)$ over $\mathbb{R}$.

Equip $V_f$ with a symmetric bilinear form $\langle \cdot, \cdot \rangle_f$ defined by
$$
\langle x_i j, y_k l \rangle_f = \begin{cases} 2 & i=j \\ -1 & i \neq j \\ 0 & i \neq j 
\end{cases}
$$

The matrix of this form is the Cartan matrix of $\Gamma$.

Ex: $A_4$ has matrix
$$
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
$$

A labeling of vertices by real numbers $x_i$ is the same as a vector $\mathbf{d} = \sum x_i x_i e_i \in V_f$.

Fix $\mathbf{d}$ and suppose $2d_i = \sum y_j d_j$. Then $\langle \mathbf{d}, \mathbf{x} \rangle = \sum d_i x_i - \sum y_j x_j = 0$.

If this is true for $\forall i$, then $\langle \mathbf{d}, \mathbf{v} \rangle = 0 \forall \mathbf{v} \in V_f$, i.e., $\mathbf{d}$ is in the kernel of $(-, -)_f$.

Note! If $\Gamma$ not connected, $\Gamma' = \bigoplus \Gamma_k$ can composed, then $V_f' \cong \bigoplus V_{e_k}$ orthogonal.

Thm: Either I $(-, -)_f$ is positive definite $\iff \Gamma'$ is Dynkin

II $(-, -)_f$ is positive semi-definite $\iff \Gamma'$ is affine Dynkin

Moreover, Ker $(-, -)$ is 1-dimensional, spanned by McKay vectors $\mathbf{w}_\mathbf{f}$.

III $(-, -)_f$ is indefinite $\iff \Gamma'$ is general type.

Recall: Indefinite means $\exists \mathbf{v}, \mathbf{w}$ s.t. $\langle \mathbf{v}, \mathbf{v} \rangle > 0, \langle \mathbf{v}, \mathbf{w} \rangle < 0$

Positive semi-definite means $\langle \mathbf{v}, \mathbf{v} \rangle > 0 \forall \mathbf{v}$, but possibly $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ for $\mathbf{v} \neq 0$.

This is the real meat behind the classification.

PP: First show affine Dynkin $\Rightarrow$ positive semi-definite with 1-D kernel.

This implies that Dynkin $\Rightarrow$ positive definite. Thus is because $V_f \subset V_f^1$, when $\Gamma \subset \Gamma'$. Let $\mathbf{v}$ be a vector by $\mathbf{0}$. Then \( \langle \mathbf{v}, \mathbf{v} \rangle > 0 \forall \mathbf{v} \in V_f \), $\Gamma$ Dynkin $\subset \Gamma'$ affine Dynkin.

But if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ then $\mathbf{v}$ is a multiple of $\mathbf{0} \Rightarrow \mathbf{v}$ not in image of $V_f$.

Then show if $\Gamma' \neq \Gamma$, if Dynkin then $\Gamma'$ is indefinite. This shows all three $\Rightarrow$ directions. But then by classification of graphs, get $\iff$ directions.
Lemma: $\Gamma$ affine Dynkin $\implies$ has semistable if $\text{Ker}(\lambda)_\Gamma = 0$.

**Proof:** We use only the existence of $\lambda \in \text{Ker}(\lambda)_\Gamma$. Let $x = \Sigma d_i x_i$. For any given $i$ let $V = \Sigma x_i x_i$. Then

$$\langle V, V \rangle = \sum_{i,j} 2x_i^2 + \sum_{i} \sum_{j} (-x_i x_j) = \sum_{i,j} \left( \frac{d_i}{dx_i} x_i^2 - x_i x_j \right)$$

Each edge appears twice in $\sum_{i,j}$. Sum over edges instead:

$$\sum_{\text{edges}} \left( \frac{d_i}{dx_i} x_i^2 - 2x_i x_j + \frac{d_i}{dx_j} x_j^2 \right) = \sum_{i,j} d_i d_j \left( \frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2 \geq 0$$

with equality iff $\frac{x_i}{d_i} = \frac{x_j}{d_j}$ for all edges $\implies V$ is a multiple of $x_i$.

Lemma: $\Gamma$ affine Dynkin $\iff$ $\Gamma'$ oriented type. Then $(\xi, \xi)_\Gamma'$ is indefinite.

**Proof:** If $\Gamma'$ has an extra edge by the vertices in $\Gamma$, then let $v = \lambda V_{\Gamma'}$ be $\Sigma v x_i$ the McKay labeling. Then $(v, v)_\Gamma' < \langle (\xi, \xi)_\Gamma' \rangle = 0$ since the edge just makes the sum more negative. But $(\xi, \xi)_\Gamma' = 2 > 0$.

If $\Gamma'$ has extra vertex $v$ connected to nothing in $\Gamma$, let $V = \lambda V + 3 v x_k$.

$$\langle V, V \rangle = \langle (\xi, \xi), 2 \rangle + 2 \cdot (\xi, \xi)_\Gamma' + 2 \lambda^2 \leq 0$$

As $\lambda > 0$, $\exists \to 0$ this is negative.

Can you prove this for $A_\infty$? $A_\infty$? $D_\infty$? $D_\infty$?...

**Rank:** Exercise uses $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{4}$ to find a vector $V$ with $\langle V, V \rangle \geq 0$.

For $\Gamma (\Gamma') = \lambda$.