Exercises Otr2 Wk 3

1. This exercise is related to Weyl's original proof of his theorem on complete reducibility.
   a) Consider: \( \text{null}(n) = \{ Y \in g(n) \mid Y + Y^* = 0 \} \). Let \( K_{std} \) denote the standard trace form on \( g(n) \), i.e., \( K_{std}(X,Y) = Tr g(X,Y) \). Pare that \( K_{std} \mid \text{null}(n) \) is negative definite.
   b) Recall that \( \text{Syl}(n) = \text{Syl}(n; \mathbb{C}) \) is simple. Deduce that \( \text{Syl}(n) \) is simple over \( \mathbb{R} \).
   c) Deduce that the Killing form on \( \text{Syl}(n) \) is negative definite. (How do \( K_{std} \) and Killing compare?)

Def: Given a complex split Lie algebra \( \mathfrak{L} \), a compact real form \( \mathfrak{g} \) of \( \mathfrak{L} \) is a Lie algebra \( \mathfrak{g} \) such that the Killing form on \( \mathfrak{g} \) is negative definite and \( \mathfrak{g} \mid \mathfrak{H} = \mathfrak{L} \).

Let \( K \) be a compact Lie group w/ \( \mathfrak{g} = \text{Lie} K \) simple. Prove that the Killing form is negative definite. (Hint: The existence of an invariant hermitian form on a rep., implies that the rep. \( \mathfrak{g} \to \mathfrak{g}(\mathbb{C}) \) factors thru \( K \to \text{SU}(n) \).)

2. Pare that every complex semisimple Lie algebra has a compact real form.
   (Hint: Read Wiki for a sketch.) (In fact, it is unique.)

Weyl proved that every compact real form is \( \text{Lie} K \) for a compact group, hence a simply connected one. This connects \( \text{Rep}_{\mathbb{R}}(g) \) to \( \text{Rep}_{\mathbb{R}}(\mathfrak{g}) \) and \( \text{Rep}_{\mathbb{R}}(K) \).

These ideas above go under the name "the criterion trick."

Humphreys

\[
\begin{align*}
\text{A}_3 & \# 8, 9, 10 \\
\text{A}_4 & \# 1, 3 \\
\text{A}_6 & \# 14, 15 \text{ acd}
\end{align*}
\]