Let $G$ be a finite group, and choose $\Delta$. Write $v\Delta w$ in $\Lambda^+$ if

$$v + \sum c_i \beta_i = w$$

for $c_i \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{L} = n^{-1} \mathcal{L}^+$. If $\mathcal{L}$ is finite, then $\mathcal{L} = \mathbb{Z}^\vee$.

**Def:** $\Delta/\mathcal{L}^+$ is a weight if $\langle \xi, \beta \rangle \in \mathbb{Z} \forall \beta \in \mathcal{L}^+ \iff \langle \xi, \beta \rangle \geq 0 \forall \beta \in \Delta$.

Then from the weight lattice $\Lambda^+ = \mathcal{L} \Lambda$.

It is $\mathcal{L} = \mathbb{Z} \cdot \{w_i\}$ where $\langle w_i, \beta \rangle = \delta_{ij}$.

If $\mathcal{L}$ is finite, then $\mathcal{L}$ is a multiset, and $\text{wts}(V) \subseteq \mathcal{L}$.

Claim: If $V$ is a rep, then $V$ is a multiset, and $\text{wts}(V) \subseteq \mathcal{L}$.

**Pf:** $V$ finite $\Rightarrow$ $V$ compact, so weakly $V$ irreducible. Let $V'$ be the spin $V$ irreducible. Let $V' = V$.

V is a subrep since $\text{ad}(L) = \mathcal{L} \cdot \mathcal{L}$, so $\mathcal{L}$ has an eigenvector. Then $V$ is a full rep of $\mathcal{L}$.

Let $\mathcal{L} = \mathbb{Z} \cdot \{w_i\}$ for $\mathcal{L}^+ = \mathbb{Z}^+ \cdot \{w_i\}$ a copy of $\mathbb{Z}^+$.

Then $V$ is a full rep of $\mathcal{L}$.

$\Rightarrow \lambda = \sum_i n_i w_i$, $\lambda \in \mathbb{Z}^+$ and $\text{wts}(V)$.

**Why?** $\mathcal{L}$ is when the $\lambda$ string appears and $\langle \lambda, \alpha \rangle = 1$.

Recall $S_\lambda = \exp(x_{\alpha}) \exp(-y_{\alpha}) \exp(x_{\alpha})$ which makes sense if $x_{\alpha}, y_{\alpha}$ are locally nilpotent.

$(S_\lambda)^2 = 1$, but $S_\lambda$ is invertible $\exp(-x_{\alpha})$ and acts on $V$ with the same $S_\lambda$.

**Immediate goals:** Classify all reps $\mathcal{L}$. Compute the multiset $\text{wts}(V)$. Every $V$ inherits $\mathcal{L}$.

**Next goals:** structures, tensor product decompositions (plethysm).
**HW Reps**

**Def:** $V$ (not nec. 1-dim.) is weight if $u \cdot \delta$.  
$V$ is hw of $h \cdot \delta$ if generated by $\lambda \in VH$ w/ $\lambda u \cdot \delta = 0$ 

(Humph: “standard cyclic”)

**Constr:** Fix any $\lambda \in \mathbb{N}^+$ (not nec. in Aut). Let $\Delta(L) = U(L) \otimes \mathbb{C}_\lambda$, where 

$m = \text{dim } \mathbb{C}_\lambda$ and $\delta$ is the 1-d $\text{SL}_2$-rep where $\gamma^+ \cdot \chi = 0$ 
$h \cdot \chi = \chi(h) \chi$.

(Rank: All 1-d reps of $\mathbb{G}$ have the form $\mathbb{C}_\chi$, since $[\delta^+, \delta^+] = (-1)$ must act by zero, and 

$\mathbb{C}_\lambda/\mathbb{C}_\chi \cong \mathbb{C}_\lambda$ where 1-d rep on given by $\lambda^+$.)

By PBW, $U(L) \otimes \mathbb{C}_\lambda \cong U(L^+) \otimes \mathbb{C}$, so $\Delta(L) = U(L^+) \otimes \mathbb{C}_\lambda$, i.e. have

a basis for $\Delta(L)$ given by $e, y_1, y_2, \ldots, y_m$  

where $y_i \cdot y_j = y_j \cdot y_i + \lambda^+ y_j$ for $i < j$.

PBW about basis not guaranteed.

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1. Then $\Delta(L)$ is hw 1-1 and if $V$ is hw 1 then $\Delta(L) \rightarrow V$.

2. If $\Delta(L) \rightarrow V$ then $V$ is hw 1. (Pf: $v_{+} = 0$ all $t \rightarrow 0$. Eval $v_{+}$.

3. Also $\text{wts } (\Delta(L)) = \begin{Bmatrix} \mu \in \mu \mathbb{N}^+ \end{Bmatrix}$, 1 appears w/mult 1, any wt w/ full mult.

4. $\Delta(L)$ has 1 max mult proper subrep $\tilde{J}$ in direct product (Any proper has no 1 in space)

5. Index: All fibres in $\text{hw} = \text{sum of mult.}$

6. Cor: Every hw 1, 2, 4: $\text{PF: } E$ by above. 1 by some arg. at $H_2$.

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Great, but when is $L$ flat? Rarely!

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**Ex:** Spore $\text{hw} \notin \mathbb{N}^+$. Then $L$ not flat. (wt($L$) $\notin \mathbb{N}^+$)

If $\lambda \in \mathbb{N}^+$ then the $\text{SL}_2$-rep $\Delta_2(\lambda) \otimes L$. 

---

**Rank: If $\text{we sl}_2(L)$ then $L$ flat, but we need above + fs. in each spot. $H_{L_1}$
What do Vermas look like? They're all the same size, just with weights shifted.

$\Delta(152)$

Ex 8.2.1

$A(0) = -6 - 4 - 2 - 0$

Ex 8.3.1

$y_1, y_2, y_3, y_4, V_4$

$\dim V[\Delta - \mu] = K(\mu) \text{ Kostant partition function}$

$= \# \{ \text{ ways to write } \mu \text{ as } \sum c_i x_i, c_i \in \mathbb{Z}_{\geq 0} \}$

$K(-3x_1 - 3x_2)$

Exercise: Find the general pattern?

$K$ is weird

$K$ is "combinatorial" (we know the size of $\Delta(\lambda)$). Why more mysteriously
Theorem: $L$ is f.d. $\iff \lambda \in \Lambda^+_\text{tot} = \{ \lambda \in \Lambda^+ \mid \langle \lambda, x \rangle \in \mathbb{Z}_{\geq 0} \}$

Moreover, if $\lambda \in \Lambda^+_\text{tot}$, then $N(\lambda) = \text{Ker} (\Delta \lambda \rightarrow \Lambda^0)$ is generated by $y^{x+1}_k V^+_k$ for $\alpha \in \Delta$.

Proof: We already know $\Rightarrow$ from the theory. Recall: $\alpha$-string thru $V^+_n = n \langle \lambda, x \rangle$

If $n \neq 0$, then no hw vec except $V^+_4$, so any subrep doesn't meet $\alpha$-string.

If $n = 0$, then only hw vec is $y^{1+1}_n V^+_1$.

$x_{\alpha}(y^{x+1}_n V^+_n) = 0$ also $x_{\beta}(y^{x+1}_n V^+_n) = 0$ for $\beta \neq \alpha$.

(or equiv. $\Delta \lambda y^{x+1}_n V^+_n \in \mathbb{Z}_{\geq 0}$, nowhere to go.)

So $y^{x+1}_n V^+_n$ generates a hw rep $v/ Lw = S_{\alpha}(\Lambda) - \alpha$. (Actually, by PBW original, it's $\Delta(\Lambda(x) - \alpha)$.)

Let $N(\lambda)$ be generated by them, $A = \Delta \lambda \Lambda/ N(\lambda)$. Enough to show that, for each $\alpha \in \Delta$, $\omega = \bigoplus f^x \omega_\alpha$-rep. If so, $W(\mathcal{W}(\omega)) \Rightarrow \omega$ for $\mathcal{W}$.

We saw this before (or very similar)! Use induction on $\omega$ to $(=) \omega$ compare, also indet. $\Rightarrow \omega$ map.

We now show that $y_\alpha$ acts mlp on $y^{x}_1 y^{x}_2 - y^{x+1}_1 V^+_n$.

We just did base case. (Actually, $n$ is your inductor, just use length of any word $y^{1+1}_n V^+_n$.) $x_\alpha$ clearly acts mlp for bold wt reasons.

So $\omega = \bigoplus f^x$ reps.

Remark: Humphreys has slick proof that the ideal quot of $\Delta \lambda$ is $\sum$ f.d. $\omega_\alpha$-reps, before he speaks the kernel $N$. This is shortcut.
Ex: $s^3 \lambda = (1,0) = \lambda_0 + \lambda_1$

$\Delta(\lambda_0) \leftarrow \Delta(\lambda) \to \lambda_1 \to 0$

\( \Delta(\lambda_0 + \lambda_1) \)

(each is individually injective!)

what remains:

maybe other stuff too, but that's all inside the convex hull of \(w_0\).

wts \(L_2 \subset \text{ Hull } (w_{-})\)!! So that's it.

\(\lambda = (1,1) = \omega_1 + \omega_2 = \alpha + \beta\)

Now one we don't know! (Well, we do)

\(\lambda = (2,0)\)

Rank: Actually, can do a lot less work. Every weight is \( \lambda \) conjugate to one in \( \text{Cone of convex hull}\).

So only need to compute these sides!

Overlap of kernels don't kick in until \(\lambda = (\lambda_0 + 1) > (\lambda_1 + 1)\beta\)

over \(\alpha\) is \(\Delta(\lambda_0 + 1) = \Delta(\beta) \Delta(\lambda_0 - 1)\).
Method to compute multiset vals ($L_x$): Look only in $G_j$ complex

$\dim \Delta(x|\alpha) - \dim \Delta(x|\alpha - \alpha) - \dim \Delta(p_1|\beta - \beta)$ for each st space $\mu$.

This is $\dim L_x[\alpha] - \dim L_x[\alpha - \alpha]$, gives all nonzero st spaces!

Does it work for $B_2$ as well? $\beta^\alpha <_p b_\beta = -2$ $\alpha^\beta <_\beta -1$

Overlap begins at $1 - <(k+1)x+1>(k+1)^\beta$

$<\beta^\alpha> <(\alpha + 1)x+2> - <(\alpha + 1)x+1>

$2 <\beta^\alpha> <\beta^\alpha> - <\beta^\alpha> - <(\alpha + 1)x+1>

Works if $1 = (m,n)$, $m n$ are $2n < m$, feel if $n < m < 2n$.

Need to add back in the overlap... yuck!

Success ex: Failure ex. (Major) exercise.

Does it work in type $A$? $\lambda = \Sigma a_i e_i$, $a_i > 0$.

Overlap at $1 - (a_i + 1)x_i - (a_i + 1)y_j$

$<x_i, x_i> <y_j, y_j> <x_k, x_k>

If odd

$\begin{pmatrix}
    a_y - a_y - 1 \\
    a_x - a_y - 1
\end{pmatrix}
\begin{pmatrix}
    a_k + \frac{1}{(a_x+1)} \\
    a_y - 1
\end{pmatrix}
$

If distinct

$-a_x - 1$  $-a_y - 1$  $a_w$  $a_y$

Works, it works. Perverse though, computing size of $\Delta$, i.e. Kostant partition function, is harder.

Want better ways.
Next major theorem: 1. Weyl dimension formula
\[ \text{dim}(L_\lambda) = \frac{\prod_{\alpha \in \delta^+} \langle \lambda + \rho, \alpha \rangle}{\prod_{\alpha \in \delta^+} \langle \rho, \alpha \rangle} \]
\[ \text{Rmk 1: Need denominator or else \text{dim}(L_\lambda) \neq 1.} \]
\[ \text{Rmk 2: } \langle \rho, \beta \rangle = 1 \quad \forall \beta \in \Delta. \text{ So if} \]
\[ h(\alpha) = k \quad \text{then } \langle \rho, \alpha \rangle = k. \]

Ex: \( s_2 \lambda = (n) \in \mathbb{Z} \) means \( n \cdot w_1 \).
\[ \langle \rho, \alpha \rangle = 1 \quad \langle \rho, \alpha \rangle = n \quad \text{dim} L_\lambda = \frac{n + 1}{2} = n + 1. \]

Ex: \( s_3 \lambda = (m, n) \in \mathbb{N}^+ \) means \( m \cdot w_1 + n \cdot w_2 \).
\[ \langle \rho, \alpha \rangle = 1 \quad \langle \rho, \alpha \rangle = m \quad \text{dim} L_\lambda = \frac{(m + 1)(m + 2)}{2} \]
\[ \langle \rho, \alpha \rangle = 2 \quad \langle \rho, \alpha \rangle = m + n \]

Ex: \( B_2 \)
\[ \lambda = (m, n) \]
\[ \langle \rho, \alpha \rangle = 1 \quad \langle \rho, \alpha \rangle = m \]
\[ \langle \rho, \alpha \rangle = 1 \quad \langle \rho, \alpha \rangle = n \]
\[ \langle \rho, \alpha \rangle = 3 \quad \langle \rho, \alpha \rangle = m + n \]
\[ \langle \rho, \alpha \rangle = 2 \quad \langle \rho, \alpha \rangle = m + n \]

maybe easy to use (2):
\[ (\alpha, \alpha) = \frac{1}{2} \quad (\alpha, \alpha) = \frac{m}{2} \quad \text{not by } \delta^+ \]
\[ \langle \rho, \alpha \rangle = 1 \quad \langle \rho, \alpha \rangle = n \]
\[ \langle \rho, \alpha \rangle = 3 \quad \langle \rho, \alpha \rangle = \frac{m + n}{2} \]
\[ \langle \rho, \alpha \rangle = 2 \quad \langle \rho, \alpha \rangle = m + n \]

\[ \text{Ex: } (1, 0) \quad \text{dim} = 4 \]
\[ \text{Ex: } (0, 1) \quad \text{dim} = 5 \]
\[ \text{Ex: } (2, 0) \quad \text{dim} = 10 \]

Rmk: \( \text{dim} L_{\lambda} = 2^{\# \Delta^+} \).

Want better - want \( \text{dim} L_{\lambda} \geq 1 \forall \mu \! \).

2) BGG Resolution (the best!)
Def: The shifted action of \( W \) on \( \mathbb{N}^+ \) is \( w \cdot \alpha = w(\alpha + \rho) - \rho \) (outside \( \rho \))

Check: Action. \( \delta \)-Check:
\[ \begin{align*}
\delta(1) & \to \delta \to \cdots
\end{align*} \]

So we proved: \( \bigoplus A_{\alpha \cdot 1} \to \Delta \to \Delta \to 0 \)
Theorem (Steinberg): $L_\lambda \otimes L_\mu = \bigoplus \nabla^w \omega^w \mu^w \nu^w \rho^w \tau^w$

Time to parse:

1. $\omega^w$ is unique if $\lambda = \mu$. Fail to exist $\lambda \neq \mu$ on a shifted weight cell.
2. $\lambda \neq \mu$ makes no sense, but will "cancel out" multiplicity for most weight space.

Many examples:

<table>
<thead>
<tr>
<th>$V_5 \otimes V_3$</th>
<th>$\bigoplus \text{ wt}(V_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 , -1 , 1 , 3$</td>
<td>$2 , 0 , 2 , 4 , 6 , 8$</td>
</tr>
</tbody>
</table>

So, mutl of $L_0$ is $1$ from $\nu = 3$, $-1$ for $\nu = 5$ and $0$. $V_4 \otimes V_3 = V_4 \otimes \nabla^w \omega^w \mu^w \nu^w \rho^w \tau^w$

$V_5 \otimes V_4 = \bigoplus \nabla^w \omega^w \mu^w \nu^w \rho^w \tau^w$
\[ L_{(5,0)} \otimes L_{(6,0)} = L_{(6,0)} \oplus L_{(4,1)} \]
\[ L_{(5,0)} \otimes L_{(5,1)} = L_{(5,1)} \oplus L_{(4,2)} \]
\[ L_{(5,0)} \otimes L_{(1,1)} = L_{(6,1)} \oplus L_{(4,2)} \oplus L_{(3,1)} \oplus L_{(5,0)} \]

Q: 2-1