Def: A topological group is a set $G$ which is both

1. a topological space
2. a group structure

and such that these structures are compatible, i.e., the maps $m: G \times G \to G$ and $i: G \to G$ and $i$ are both continuous. (A group has a third structure map, $u: G \to G$ which is neither continuous nor inverse.)

A morphism of top. g. ps. is a map $\phi: G \to H$ which is both

1. continuous
2. a group homomorphism.

Rem: Note that $\log: G \to G$ and $\exp: G \to G$ are homeomorphisms and composition by $g$ is continuous.

Ex: 1. Any group with discrete topology. All finite groups are considered as top. g. ps. with this topology. (Their idea of homomorphism is unchanged.)

2. Any group with indiscrete topology. (This is not what you want.) The maps induce $\mathbb{R}^n$ etc.

3. $\text{GL}(n; \mathbb{R})$ and $\text{GL}(n; \mathbb{C})$, and all Lie groups (soon).

4. Profinite groups and other completions (see Exercise, they are important in many contexts, but their rep theory is a whole field related to number theory.)

Def: A (continuous) (real) repn of a topological group is an action of $G$ on a topological vector space $V$ over either $\mathbb{R}$ or $\mathbb{C}$ (or some other topological field $F$) such that the corresponding map $\rho: G \to \text{GL}(V)$ is continuous. Here, $\text{GL}(V)$ is a top gp via the identification $GL(V) \cong GL(n; F)$ coming from a choice of basis. (Exercise: why no dependence on choice of basis?)

Exercise: Find a non-continuous repn of $(\mathbb{R}, +)$.

Other categories defined in similar ways.

Def: A Lie group is simultaneously a group and a smooth $\mathbb{R}$-manifold. Morphisms (smooth) repr. one to one.

A complex Lie group

Ex: Finite g. ps. are $\mathbb{Z}$-dim Lie groups.

Ex: $S^1$ is a real Lie gp. $\mathbb{C}^\times$ is a $\mathbb{C}$-Lie gp.

A 1D $\mathbb{C}$ repn of $S^1$ is a map $S^1 \to \text{GL}(1; \mathbb{C}) \cong \mathbb{C}^\times$. 
The group $G$ of $C^*$ is holomorphic if it is holomorphic. So the map $C^* \to GL(4)$ is smooth but not holomorphic.

We can use topological concepts to study top groups. Connectedness, compactness, etc.

(Any Lie group is locally path connected so connected $\Rightarrow$ path connected. I will typically assume l.p.c.)

Prop: Let $G$ be a (l.p.c.) top gp, and let $G_0$ denote the connected component of $1 \in G$.

Then $G_0 \triangleleft G$, and $G/G_0$ is a discrete group called the quotient group, $C_0 = \frac{G}{G_0}$.

$C_1 \cdot C_2 = C_3$ iff $\exists g \in G, C_3 = g \cdot C_2 \cdot g^{-1} = C_1$.

Proof: Suppose $g \in G_0$. Then $\exists$ path $p: I \to G$ s.t. $p(0) = e, p(1) = g$.

Then $p(t)p^{-1}(1)$ is a path from $1$ to $g$. It is continuous.

Suppose $g \in G_0$ and $k$ is arbitrary. Then $kgk^{-1}$ is a path from $1$ to $kgk^{-1}$.

Ex: $G = O(n)$. Then $G_0 = SO(n)$ and $C_0 = \mathbb{Z}/2\mathbb{Z}$.

(Clearly $O/\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ via $det: O \to \mathbb{R}^*$. So enough to show $SO(n)$ connected.)

Ex: Suppose $G$ connected, $H \subseteq G$ is discrete and normal. Then $H \cong Z(G)$.

Ex: Exercise.

Thus: $G$ a top gp, then $\tilde{G}$ (the universal cover) has a natural structure of a top gp s.t.

$\tilde{G} \to G$ is a morphism.

The idea is similar to the proof above - take a path $p$, translate it by $\tilde{g}$, and lift to endpoint. Now check it is independent of path, and satisfies the group axioms...

(I'm thinking about parts in $\tilde{G}$ as equiv classes of paths to points in $G$, from the base point $1$. )
Cor: \( G \) connected \( \Rightarrow \) \( \pi_1(G) \) is abelian,

Pf: \( G \) is connected \( \Rightarrow \) \( G \to \tilde{G} \to G \) where \( \tilde{G} \) is the covering group of \( G \) such that \( \tilde{G}/\pi \tilde{G} \) is isomorphic to \( \pi_1(G) \).

Notes: If \( H \subseteq \pi(G) \) and is discrete, then \( G \to G/H \) is a universal covering map (in the case where \( G \) is a simply connected Lie group).

Ex: \( \mathbb{Z}(SU(n)) \cong \mathbb{Z}^{n-1} \) where \( SU(n) \) is the special unitary group.

- [Rmk] If \( G \) is compact then any discrete subgroup is finite (true for all compact groups).
- [Rmk] If \( G \) is simply connected and \( \pi_1(G) \) is trivial, then \( G/\pi(G) \cong G \) but \( \tilde{G}/\pi(G) \cong \pi(G) \).

Galois theory says:

\[ \left\{ \text{subgroups of } \pi_1(G) \right\} \leftrightarrow \left\{ \text{coverings } \tilde{G} \to G \right\} \]

Covering maps will be really important. For Lie groups, \( \pi_1 G \cong ag \) is the Lie algebra. Any covering map \( G \to \tilde{G} \) will induce a local diffeomorphism at \( 1 \) and thus \( \tilde{G} \to G \cong \pi_1 G \) is an iso!!

Compactness will be of huge importance as well. Ex: \( S^1 \) is compact, \( \mathbb{C}^* \) is not.

Rmk: \( G \) compact \( \Rightarrow \chi(G) = 0 \)

Pf: Let's denote the fixed point set \( X \subset \mathbb{R}^X \).

If \( f: X \to X \) has a fixed point, then \( f^k \) has no fixed points for any \( k \).

Ex: \( SU(2) \cong S^3 \) \( \cong \mathbb{R}P^2 \) and \( SO(3) \cong \mathbb{R}P^2 \) so there is no possible group structure on \( S^2 \).

The main reason compact groups are so important is

Rmk: \( G \) compact \( \Rightarrow \) Rep \( G \) is semisimple.

Idea: Everything you do for finite groups can be done (using integration) for compact groups!