

# Interlude: Categorical Nonsense

(1)

Recall: An object  $F$  in a cat.  $\mathcal{C}$  is final if  $\forall X \in \mathcal{C} \exists! X \rightarrow F$

Easy: Two final objects are isom (via unique isom).

Def: A zero object is both initial and final.

Ex: In sets,  $\emptyset$  initial,  $\{*\}$  final, no zero.

In  $\text{Sets}_*$ ,  $\{*\}$  is both initial + final.

Ex: Vect,  $R\text{-mod}$ , other cats w/ forgetful functor to  $\text{Sets}_*$

If zero exists,  $\forall X, Y \in \mathcal{C} \exists! X \rightarrow \text{zero} \rightarrow Y$  the zero morphism.  $0 \circ f = 0 = g \circ 0$ .

Recall: Products, coproducts.

$$\text{Hom}(W, \prod X_i) = \prod \text{Hom}(W, X_i)$$

$$\text{Hom}(\coprod X_i, W) = \prod \text{Hom}(X_i, W)$$

Convention:  $\prod_{\emptyset} X_i =$  initial object,  $\prod_{\emptyset} X_i =$  final object.

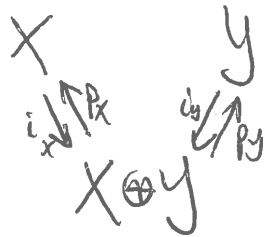
Note: If  $X \times Y$  exists  $\forall X, Y$  then finite products exist but infinite might not.

Ex:  $R\text{-mod.}$  vs.  $R\text{-Mod.}$

Def:  $X \oplus Y$  is BOTH  $X \times Y$  and  $X \amalg Y$  (if it exists)

Ex: Sets  $X \times Y \neq X \amalg Y$   
Vect  $X \times Y \cong X \amalg Y =: X \oplus Y$

Explicitly



s.t.  $\forall Z, \textcircled{*} \text{Hom}(X \oplus Y, Z) \cong \text{Hom}(X, Z) \times \text{Hom}(Y, Z)$   
 $\textcircled{**} \text{Hom}(Z, X \oplus Y) \cong \text{Hom}(Z, X) \times \text{Hom}(Z, Y)$

$$\psi \mapsto (p_X \circ \psi, p_Y \circ \psi)$$

ie direct sum has 4 maps and  $\textcircled{*}, \textcircled{**}$  are invertible.

What maps exist  $X \rightarrow X \times Y$ ? Same as a map  $X \rightarrow X$  and a map  $X \rightarrow Y$  (2)

Example: Given  $f: X \rightarrow Y$  have  $\text{Graph } f: X \rightarrow X \times Y$ , corresponds to  $(\text{id}_X, f)$ .

In general there might not be any maps  $X \rightarrow Y$ ! If zero object exists, then

get  $i_X: X \rightarrow X \times Y$  corresponds to  $(\text{id}_X, 0)$

$i_Y: Y \rightarrow X \times Y$  for  $(0, \text{id}_Y)$

$\rightsquigarrow X \amalg Y \rightarrow X \times Y$  for  $(i_X, i_Y)$  canonical map. Might not be isom!

~~Example: If  $X$  and  $Y$  are objects in  $\mathcal{C}$ , then  $X \times Y$  exists!~~

Def: A preadditive category is a cat. where

- $\text{Hom}(X, Y)$  is an abelian gp
- Composition is bilinear

$\Rightarrow \exists X \circ Y$  (even if no zero object)

Exercise: If zero exists,  $X \rightarrow Z \rightarrow Y$  equal  $X \circ Y$

Note: This is NOT the same as cat. enriched over  $\text{AbSp}$

•  $X \circ X \neq X \xrightarrow{\text{id}_X} X$

• bilinear  $A \times A' \rightarrow B$  is NOT homomorphism

~~Implications for  $\oplus$ :~~

~~In a preadd cat~~

~~Thm:~~

~~(1)  $A \times X = \text{id}_X$   
 $P_Y \times Y = \text{id}_Y$~~

~~(2)  $A \times Y = 0$   
 $P_Y \times X = 0$~~

~~(3)  $\text{id}_D = i_X \times P_X + i_Y \times P_Y$~~

~~$D \triangleq X \oplus Y$  iff  
Morever (3)  $\neq$  (1)  $\neq$  (2)  
Products and coproducts are direct sums!~~

Implications for  $\oplus$  Think: In vect  $V \oplus W \cong V$  have explicit formulas. (3)

Thm 1: In preadd. cat, products are coproducts and coproducts are direct sums!

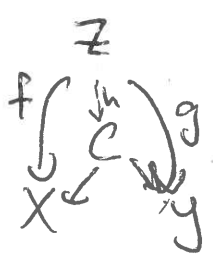
PF: Let  $\begin{array}{ccc} X & & Y \\ & \searrow i_x & \swarrow i_y \\ & C & \\ & \swarrow p_x & \searrow p_y \\ X & & Y \end{array}$  be a coproduct. Define  $p_x: C \rightarrow X$  corresponding to  $(id_X, 0)$  and sm.  $p_y: C \rightarrow Y$  for  $(0, id_Y)$

by construction, ①  $p_x i_x = id_X$   
 $p_y i_y = id_Y$   
 ②  $p_x i_y = 0$   
 $p_y i_x = 0$

③  $i_x p_x + i_y p_y = id_C$

Let  $e := i_x p_x + i_y p_y$ . Then  $e i_x = i_x p_x i_x + i_y p_y i_x = i_x \stackrel{!-lft}{=} id_X$   
 $e i_y = i_y p_y i_y = id_Y$   
 (addition, yay)

Now we can prove product property.



Think: In vect  
 $h = i_x \circ f + i_y \circ g$   
 i.e.  $h(z) = (f(z), g(z))$

Let  $h = i_x \circ f + i_y \circ g$ .

Then  $p_x \circ h = f$   $p_y \circ h = g$ , it's a lift.

If  $h'$  another lift,  $h - h'$  a lift of  $(0, 0)$ . WTS  $0$  is unique lift of  $(0, 0)$ .

If  $p_x \circ h = 0$   $p_y \circ h = 0$  then  $h = (i_x p_x + i_y p_y) h = 0$ .

Thm 2: Data of direct sum  $\iff$  object  $C$  w/ maps  $i_x, i_y, p_x, p_y$  satisfying ①, ②, ③. Also, ①, ③  $\implies$  ②

PF: ③  $p_y i_x = p_y (i_x p_x + i_y p_y) i_x$   
 $= p_y i_x + p_y i_x$   
 $\implies p_y i_x = 0$

~~Products and direct sums are the same~~

Note: Infinite products and coproducts are different.

In vect,  $\bigoplus_{\infty}$  is coproduct, not product

Thm 3: Suppose  $\textcircled{4}$  exists. Given  $f, g: X \rightarrow Y$  then  $\textcircled{4}$   
 $f+g$  equals  $X \xrightarrow{(id_X, id_X)} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{(id_Y, id_Y)} Y$ .

PF: Exercise.

Thm 4:  $F: A \rightarrow B$  functor w/  $B$  preadd,  $A$  preadd w/ finite  $\oplus$  (and zero)

TFAE 1)  $F(f+g) = F(f) + F(g)$

2)  $F\left(\begin{array}{ccc} & X \oplus Y & \\ \nearrow & & \searrow \\ X & & Y \end{array}\right)$  is a direct sum in  $B$ ,  
 diagram

PF: 2)  $\Rightarrow$  1) comes from Thm 3, describing addition w/ direct sum.  
 1)  $\Rightarrow$  2) comes from Thm 2, b/c  $F$  preserves condition ① ② ③.

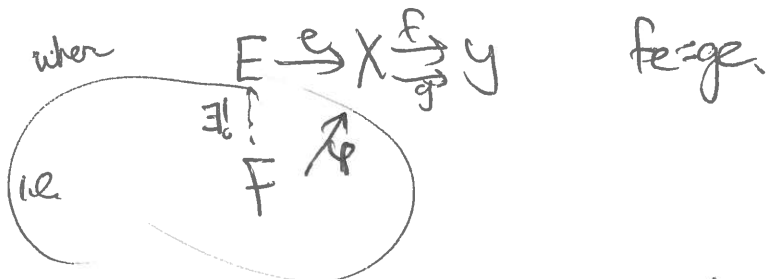
Such a functor is called additive.

Def: An additive cat  $A$  is preadd + has finite  $\oplus$  (and zero)

By default, we only care about additive functors b/w additive cats.

Kernels + Cokernels

Def: Given  $X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$ , an equalizer is a final object  $(E, e)$  in category



Similarly, coequalizer is initial w/  $X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y \rightrightarrows C \quad cf = cg$

Ex: In Set or Top,  $E = \{x \in X \mid f(x) = g(x)\}$  and  $e$  is inclusion ⑤  
 $C = \{y \mid \dots\}$  where  $f(x) = g(x)$ .  $C$  is quotient

In Grps  $E = \{x \in X \mid f(x) = g(x)\}$ , a subgroup  
 $C$  need not exist, since  $\{f(x)g(x)^{-1}\}$  need not be normal!

In AbGrps,  $C$  exists.

Recall:  $E \hookrightarrow X$  is called monic iff 
$$F \begin{matrix} \xrightarrow{h} \\ \xrightarrow{k} \end{matrix} E \xrightarrow{e} X \quad \text{else } k \Rightarrow h = k$$

ie  $e \circ (-)$  is injective  $\text{Hom}(F, E) \rightarrow \text{Hom}(F, X) \quad \forall F$ .

For Set, Vect, etc, monic  $\Leftrightarrow$  injective.

Similarly,  $X \hookrightarrow C$  is epic if  $(-)_\circ c$  is injective from  $\text{Hom}(C, F) \rightarrow \text{Hom}(X, F)$

For Set, Vect, etc epic  $\Leftrightarrow$  surjective

Exercise: equalizers are monic, coeq are epic.

~~...~~

Def: Suppose  $\mathcal{C}$  has zero. A kernel  $(\text{Ker} f, \text{ker} f)$  is the equalizer of  $X \rightrightarrows Y$ , and  $(\text{Coker} f, \text{coker} f)$  is coequalizer.

(if it exists)

(No Mention of elements!)

Easy: In additive cat,  $\text{Ker}(f-g) = \text{Eq}(f, g)$  so kernel exist  $\Rightarrow$  equalizer exist.

Note: Not every monic map is a kernel.

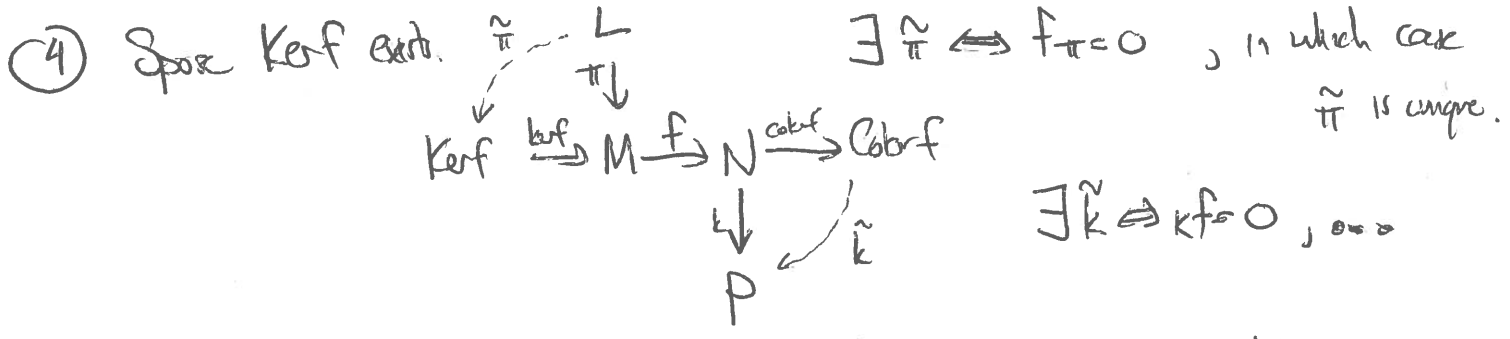
Ex: inclusion of non-normal subgroup in Grps.

Stupid examples:  $\text{Ker} f$  does not exist in  $\text{EucDimVect}$ .

Many expected properties hold, prove w/ univ. props not elements

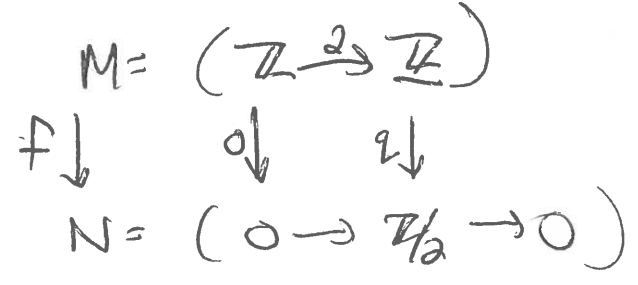
Thm:  $\mathcal{A}$  additive.

- ①  $M \in \mathcal{O} \iff \text{id}_M = 0 \iff \text{End}(M) = 0$
- ②  $M \xrightarrow{f} N$  monic  $\iff 0 = \text{Ker} f$  ; epic then  $0 \cong \text{Coker} f$
- ③  $\text{Ker}(M \xrightarrow{g} N) \cong (M \xrightarrow{\text{id}} M) \cong \text{Coker}(N \xrightarrow{g} M)$



BUT :  $M \xrightarrow{f} N$  both mono+epi does NOT imply  $f$  is isom!!!

Ex 1:  $\mathcal{C} =$  homotopy cat of  $\mathbb{Z}$ -modules.



Claim:  $0 \cong \text{Coker} f \cong \text{Ker} f$

(In  $\text{Ch}(\mathbb{Z})$ ,  $\text{Coker} f \cong 0$  and  $\text{Ker} f = (\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \cong (\mathbb{Z} \xrightarrow{2} \mathbb{Z})$   
 in  $\text{K}(\mathbb{Z})$ ,  $\text{ker} f \cong 0$  )

But  $f$  has no inverse

Ex 2: Come up w/ stupid ad hoc cat. where it fails.

Def: An abelian cat is an additive cat where

- ①  $\text{Ker} f, \text{Coker} f$  exist  $\forall f$
- ② Every monic  $\cong$  a kernel
- Every epic  $\cong$  a cokernel

Prop: In ab. cat,  $f \text{ mono+epi} \Rightarrow f \text{ isom.}$

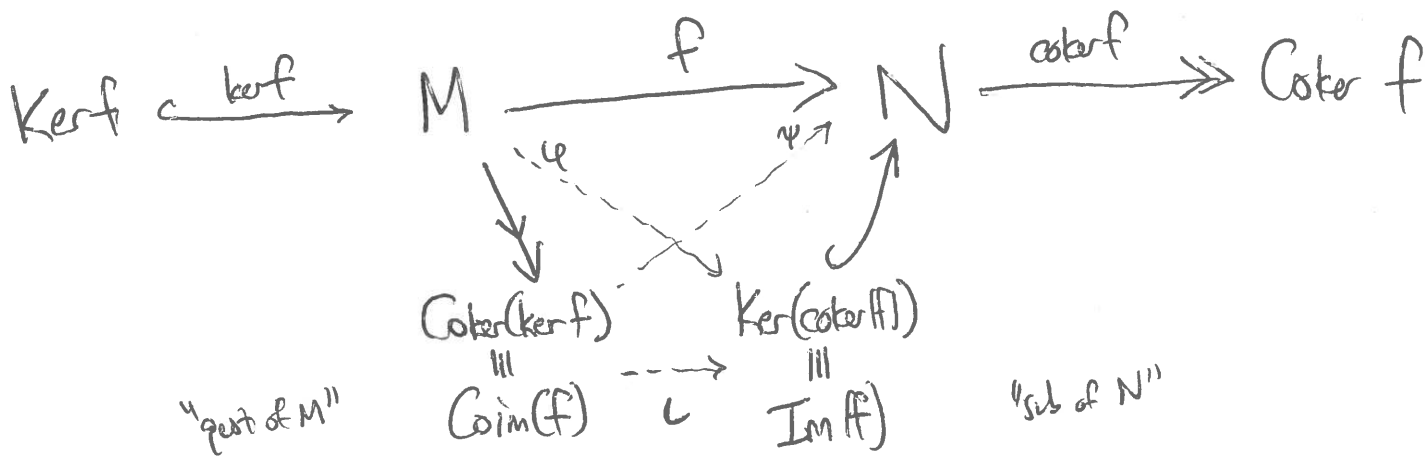
(7)

PF:  $f \text{ mono} \Rightarrow f = \ker g$  for some  $g$ .  $M \xrightarrow{f} N \xrightarrow{g} P$ .

But  $f \text{ epi} \Rightarrow g = 0$ .  $\ker(0)$  is isom.  $\square$

Exercise:  $g \text{ mono} \Rightarrow g = \ker(\text{coker } g)$  Cor: Every kernel is kernel of an epi.

The big nasty  $\mathcal{A}$  additive.



•  $\varphi, \psi$  exist by lifting (eg,  $\text{coker } f \circ f = 0$  so  $\psi$  exists)

•  $\text{coker } f \circ \psi = 0$ ,  $\varphi \circ \ker f = 0$ . (eg,  $\text{coker } f \circ (f \circ \ker f) = 0$  so has UNIQUE descent, both  $0$  and  $\varphi \circ \ker f$  are descents.)

• so  $i$  exists, descent of both  $\varphi$  and  $\psi$ . (descent unique)

•  $i$  is both epi and mono. (yucks...) (wrt  $\varphi$  epi,  $\psi$  mono)

But  $\hookrightarrow$  need not be isom (see earlier example is  $K(\mathbb{Z})$ .)

ie.  $M / \ker f \not\cong \text{Im } f$  in general.

But for abelian cat,  $\hookrightarrow$  is isom,  $\text{Coim}(f) \cong \text{Im}(f)$  canonically. (equivalent defn of abelian)

Cori In ab cat,  $M \xrightarrow{f} N$  mono-epi decomp. (8)

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow & \nearrow \\ & \text{Inf} & \end{array}$$

Def In ab cat, a subobject of  $M$  is a mono map  $S \hookrightarrow M$ .

Two are equivalent if 
$$\begin{array}{ccc} S & \xrightarrow{f} & M \\ \uparrow & \searrow & \nearrow \\ S' & \xrightarrow{g} & M \end{array}$$

No notion of element

A quotient is ...

Cori  $\{\text{subobjects}\} / \sim \iff \{\text{quotients}\} / \sim$

$$\begin{array}{ccc} (S \hookrightarrow M) & \longmapsto & \text{Coker } s \\ (\text{Ker } g) & \longleftarrow & M \xrightarrow{g} Q \end{array}$$

Def:  $A \xrightarrow{f} B \xrightarrow{g} C$  is exact at B if

- 1)  $gf = 0$
- 2)  $\text{Ker } g = \text{Im } f$  as subobjects of  $B$ .

Note: Always  $\text{Im } f \leftrightarrow \text{Ker } g$  by ① so can construct

"  $\frac{\text{Ker } g}{\text{Im } f} \cong \text{Ker}(\text{Cof } f \rightarrow \text{Coker } g)$  which measures failure of exactness.

$\text{Coker}(\text{Im } f \rightarrow \text{Ker } g)$

Fine, but: HOW TO PROVE SNAKES LEMMA w/o elements!

Approach 1: Freyd ('64): A small ~~category~~ <sup>abelian</sup> cat.  $\exists$  additive

$F: A \rightarrow R\text{-Mod}$  for some (co)ring  $R$ .

Mitchell ('65):  $F$  can be made full + faithful + exact.

full: if you construct  $P$  in  $R\text{-Mod}$ , you get it in  $A$ .

(Not essentially surjective!)  
 $A$  need not have projectives...  
 like  $\mathbb{Q}$ -mod- $\mathbb{Z}$



Ok, that's cheating.

Method 2: Use members, not elements.

Def:  $(A \in \mathcal{A})$  A premember of  $A$  is a map  $M \twoheadrightarrow A$  for some  $M \in \mathcal{A}$   
(abduction)

(Think: if  $A \in \mathcal{R}\text{-Mod}$ ,  $a \in A$ , have a map  $\mathbb{R} \rightarrow A$   
 $1 \mapsto a$ )

Two premembers are equiv ( $x \equiv y$ ) if  $\exists P \begin{matrix} u \rightarrow M \xrightarrow{x} A \\ v \rightarrow N \xrightarrow{y} A \end{matrix}$  (and both are equiv to  $x \cup y: P \rightarrow A$ )

Claim 1: Equiv reln. A member is an equiv class

Note:  $M \twoheadrightarrow A$  equiv to  $M \circ N \xrightarrow{p} M \twoheadrightarrow A$ . ~~So any two members have a common source.~~

~~Why can we talk about members of  $\mathcal{A}$ ?~~

Note:  $M \twoheadrightarrow A$  equiv to  $M \xrightarrow{id} M \xrightarrow{x} A$   
 $x \rightarrow Im x \rightarrow A$   
 (epimono decomp)  
 so every member has a unique subobject, (up to equiv)

(Unique b/c  $Im x = Im x \cup y$  since  $x \cup y \in \text{epi}$ )  
 $\because Im y \cup x = Im y$

Members are a more convenient way to discuss subobjects, b/c

1) Any two members can be given a common source.

$$(M \twoheadrightarrow A) \equiv (M \circ N \xrightarrow{p} M \twoheadrightarrow A)$$

2) Members are functorial.  $A \xrightarrow{f} B$   $f \circ (-)$  sends  $Mem(A) \rightarrow Mem(B)$   
 $\begin{matrix} A & \xrightarrow{f} & B \\ x \uparrow & & \uparrow y \\ & M & \end{matrix}$

3)  $f$  mono  $\iff \forall x \in Mem(A), fx = 0 \implies x = 0$ .  
 $f$  epic  $\iff \forall y \in Mem(B) \exists x \in Mem(A) w/ f(x) \equiv y$ .  
(Not same as saying  $f$  injective on members!)

$A \xrightarrow{f} B \xrightarrow{g} C$  exact  $\iff gf = 0$  and  $\forall x \in Mem(B) w/ gx = 0, \exists y \in Mem(A) w/ f(y) = x$ .

Now you can prove 5-lemma w/ Members rather than elements!

(10)

Complexation: Despite ① you can't add members!! Can add pre-mem w/ same source but doesn't preserve eqn.

Ex:  $A = \text{Vect}$   $A = \mathbb{C}^2$   $\text{Mem}(A) = \{0, \text{all lines in } \mathbb{C}^2, \mathbb{C}^2\}$  here.

Ex:  $A = \mathbb{C}$

$\mathbb{C} \xrightarrow{1} \mathbb{C}$  is a member

$\mathbb{C} \xrightarrow{-1} \mathbb{C}$  is a member

$\mathbb{C} \xrightarrow{2} \mathbb{C}$  is a member

All eqn's

$H1 = \mathbb{Z}$

$H(-1) = 0$

← NOT equal

Method 3: Members are great for checking properties of maps (inj, surj) but not for constructing maps. E.g. for Snake lemma can construct

" $\text{Mem}(f)$ ":  $\text{Mem}(\text{Ker } f) \rightarrow \text{Mem}(\text{Coker } f)$ , but we need this to come from  $\mathcal{D}$ .

A "better" proof of Snake lemma: use fiber squares.

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\tilde{f}} & Y \\ \tilde{g} \downarrow \Gamma & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

$X \times_Z Y$  is final object in cat

$$\begin{array}{ccc} \bullet & \rightarrow & Y \\ \downarrow \circlearrowleft & & \downarrow \\ X & \rightarrow & Z \end{array}$$

pullback

Exists in abelian cat as follows:

$$\left( X \times_Z Y \rightarrow X \oplus Y \xrightarrow[\text{g} \circ \text{p}_Y]{\text{f} \circ \text{p}_X} Z \right)$$

equalizer

Properties: •  $g$  surjective  $\Rightarrow \tilde{g}$  surjective

•  $g$  injective  $\Rightarrow \tilde{g}$  injective

Moreover,  $\text{Ker } g \cong \text{Ker } \tilde{g}$  naturally.

Similarly, have pushout

$$\begin{array}{ccc} Z & \rightarrow & Y \\ \downarrow & \searrow & \downarrow \\ X & \rightarrow & X \times_Z Y \end{array}$$

via  $\text{Coeq}(Z \rightrightarrows X \oplus Y)$

$$\begin{array}{ccccccc}
 0 & \rightarrow & A_1 & \rightarrow & P & \rightarrow & \text{Ker } f_3 & \rightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & 0 \\
 & & f_{12} & & f_{23} & & f_{34} & & \\
 0 & \rightarrow & B_1 & \rightarrow & B_2 & \rightarrow & B_3 & \rightarrow & 0 \\
 & & \downarrow & & \parallel & & \parallel & & \\
 0 & \rightarrow & \text{Coker } f_1 & \rightarrow & Q & \rightarrow & B_3 & \rightarrow & 0
 \end{array}$$

Exact rows but not columns (11)

Let  $\tilde{\delta}: P \rightarrow Q$  be  $P \rightarrow A_2 \rightarrow B_2 \rightarrow Q$ .

(Use naturality/diagram chase to) prove

$$\begin{array}{ccc}
 \begin{array}{c} P \\ \downarrow \tilde{\delta} \\ Q \end{array} & \rightarrow & B_3 = 0 \\
 & & \\
 A_1 & \rightarrow & P \\
 & & \downarrow = 0 \\
 & & Q
 \end{array}$$

The descent shows that  $\tilde{\delta}$  induces  $\text{Ker } f_3 \xrightarrow{\tilde{\delta}} \text{Coker } f_1$   
 $\text{Coker}(A_1 \rightarrow P) \quad \text{Ker}(Q \rightarrow B_3)$

We'll use pushouts/pullbacks many times. It's a trick for finding morphisms of SES

Given

$$\begin{array}{ccccccc}
 0 & \rightarrow & A_1 & \rightarrow & P & \rightarrow & C_3 & \rightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & 0
 \end{array}$$

want to fill it in.