Interlude: Categorical Nonsense

Recall: An object F in a cat $\mathcal{C}$ is *final* if $\forall X \in \mathcal{C}\ \exists!\ X \to F$.

Easy: Two final objects are isom (via unique isom).

Def: A *zero object* is both initial and final.

Ex: In sets, $\emptyset$ initial, $\{*\}$ final, no zero.
In $Sets_*$, $\{*\}$ is both initial + final.

Ex: Vect, R-mod, other cats w/ forgetful functor to $Sets_*$

If zero exists, $\forall X, Y \in \mathcal{C}\ \exists!\ X \to \text{zero} \to Y$ the zero morphism. $0 \circ f = 0 = g \circ 0$.

Recall: Products, coproducts.

$$Hom(W, \prod X_i) = \prod Hom(W, X_i)$$
$$Hom(\coprod X_i, W) = \coprod Hom(X_i, W)$$

Convention: $\coprod_{\emptyset} X_i =$ initial object, $\prod_{\emptyset} X_i =$ final object.

Note: If $X \times Y$ exists $\forall X, Y$ then finite products exist but infinite might not.

Ex: R-mod. vs. R-Mod.

Def: $X \oplus Y$ is BOTH $X \times Y$ and $X \amalg Y$ (if it exists)

Ex:
Sets $\quad X \times Y \neq X \amalg Y$
Vect $\quad X \times Y \cong X \amalg Y =: X \oplus Y$

Explicitly

$$\begin{tikzcd}
X \arrow[dr, shift right, "i_X"'] & & Y \arrow[dl, shift left, "i_Y"] \\
& X \oplus Y \arrow[ul, shift right, "p_X"'] \arrow[ur, shift left, "p_Y"] &
\end{tikzcd}$$

s.t. $\forall Z$,

$$\varphi \mapsto (\varphi \circ i_X, \varphi \circ i_Y)$$
$$(*)\quad Hom(X \oplus Y, Z) \cong Hom(X, Z) \times Hom(Y, Z)$$
$$(**)\quad Hom(Z, X \oplus Y) \cong Hom(Z, X) \times Hom(Z, Y)$$
$$\psi \mapsto (p_X \circ \psi, p_Y \circ \psi)$$

i.e. direct sum has 4 maps and $(*)$, $(**)$ are invertible.

What maps exist $X \to X \times Y$? Same as a map $X \to X$ and a map $X \to Y$.

Example: Given $f: X \to Y$ have $\mathrm{Graph}_f : X \to X \times Y$, correspond to (id_X, f).

In general there might not be any maps $X \to Y$! If zero object exists, then get

$$i_X : X \to X \times Y \quad \text{correspond to} \quad (id_X, 0)$$
$$i_Y : Y \to X \times Y \quad \text{for} \quad (0, id_Y)$$

$\rightsquigarrow$ $X \amalg Y \to X \times Y$ for (i_X, i_Y) canonical map. Might not be isom!

~~Example: If ∃ an isomorphism ... $X \oplus Y$ exists!~~

Def: A preadditive category is a cat. where

- $\mathrm{Hom}(X, Y)$ is an abelian gp
- Composition is bilinear

$\Rightarrow \exists\ X \xrightarrow{0} Y$ (even if no zero object)

Exercise: If zero exists, $X \to Z \to Y$ equals $X \xrightarrow{0} Y$

Note: This is NOT the same as cat. enriched over AbGp

- $X \xrightarrow{0} X \neq X \xrightarrow{id_X} X$
- bilinear $A \times A' \to B$ is NOT homomorphism

~~Implications for $\oplus$:~~ ~~In a pre-add cat~~

~~Thm: $X \underset{p_X}{\overset{i_X}{\rightleftarrows}} D \underset{i_Y}{\overset{p_Y}{\rightleftarrows}} Y$ then $D \cong X \oplus Y$ iff~~

~~① $p_X i_X = id_X$, $p_Y i_Y = id_Y$ ② $p_X i_Y = 0$, $p_Y i_X = 0$ ③ $id_D = i_X p_X + i_Y p_Y$.~~

~~Moreover ③+① ⟹ ②~~

~~2) (Finite) Products and coproducts are direct sums!~~

Implications for $\oplus$ | Think: In vect, $V \oplus W \rightleftarrows V$ have explicit formulas.

Thm 1: In preadd. cat, products are coproducts are direct sums!

Pf: Let
$$\begin{tikzcd} X \arrow[dr, "i_X"'] & & Y \arrow[dl, "i_Y"] \\ & C \arrow[dl, dashed, "p_X"'] \arrow[dr, dashed, "p_Y"] & \\ X & & Y \end{tikzcd}$$
be a coproduct. Define $p_X : C \to X$ corresponding to $(\mathrm{id}_X, 0)$ and sim. $p_Y : C \to Y$ for $(0, \mathrm{id}_Y)$

by construction, ① $p_X i_X = \mathrm{id}_X$ ② $p_X i_Y = 0$ $\boxed{③\ i_X p_X + i_Y p_Y = \mathrm{id}_C}$

$p_Y i_Y = \mathrm{id}_Y$ $p_Y i_X = 0$

Let $e := i_X p_X + i_Y p_Y$. Then $e \circ i_X = i_X p_X i_X + i_Y p_Y i_X = i_X$ $\xRightarrow{!\text{-lift}}$ $e = \mathrm{id}_C$.

(addition, yay) $e \circ i_Y = i_Y$

Now we can prove product property.
$$\begin{tikzcd} & Z \arrow[dl, bend right, "f"'] \arrow[d, "h"] \arrow[dr, bend left, "g"] & \\ X & C \arrow[l] \arrow[r] & Y \end{tikzcd}$$

(Think: In VS, $h = i_X \circ f + i_Y \circ g$ i.e. $h(z) = (f(z), g(z))$)

Let $h := i_X \circ f + i_Y \circ g$.

Then $p_X \circ h = f$ $p_Y \circ h = g$, it's a lift.

If h' another lift, $h - h'$ a lift of $(0,0)$. WTS 0 is unique lift of $(0,0)$.

If $p_X \circ h = 0$ $p_Y \circ h = 0$ then $h \underset{③}{=} (i_X p_X + i_Y p_Y) h = 0$.

Thm 2: Data of direct sum $\Longleftrightarrow$ object C w/ maps i_X, i_Y, p_X, p_Y satisfying ①,②,③. Also, ①,③ $\Rightarrow$ ②

Pf: $p_Y i_X \overset{③}{=} p_Y (i_X p_X) i_X + p_Y (i_Y p_Y) i_X \underset{①}{=} p_Y i_X + p_Y i_X$

Note: Infinite products and coproducts are distinct. In vect, $\bigoplus_{\infty}$ is coproduct, not product

Thm 3: Suppose $\oplus$ exists. Given $f, g : X \to Y$ then

$f+g$ equals $X \xrightarrow{(\mathrm{id}_X, \mathrm{id}_X)} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{(\mathrm{id}_Y, \mathrm{id}_Y)} Y$.

Pf: Exercise.

Thm 4: $F : \mathcal{A} \to \mathcal{B}$ functor w/ $\mathcal{B}$ preadd, $\mathcal{A}$ preadd w/ finite $\oplus$ (and zero).

TFAE

1) $F(f+g) = F(f) + F(g)$

2) $F\left(\begin{tikzcd}[column sep=small] & X \oplus Y \arrow[dr, shift left] & \\ X \arrow[ur, shift left] & & Y \arrow[ul, shift left] \end{tikzcd}\right)$ is a direct sum diagram in $\mathcal{B}$.

Pf: 2) $\Rightarrow$ 1) comes from Thm 3, describing addition using direct sum.
1) $\Rightarrow$ 2) comes from Thm 2, b/c F preserves condition ① ② ③.

Such a functor is called <u>additive</u>.

Def: An <u>additive cat</u> $\mathcal{A}$ is preadd + has finite $\oplus$ (and zero).

By default, we only care about additive functors b/w additive cats.

Kernels + Cokernels

Def: Given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, an <u>equalizer</u> is a final object (E, e) in category

where $E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ $fe = ge$.

i.e.
$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ F \arrow[u, dashed, "\exists!"] \arrow[ur, "\varphi"'] & & \end{tikzcd}$$

Similarly, <u>coequalizer</u> is initial w/ $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{c} C$ $cf = cg$

Ex: In Set or Top, $E = \{x \in X \mid f(x) = g(x)\}$ and e is inclusion
$C = Y/\sim$ where $f(x) \sim g(x)$. C is quotient

In Gps $E = \{x \in X \mid f(x) = g(x)\}$, a subgp
C need not exist, since $\{f(x)g(x)^{-1}\}$ need not be normal!

In AbGps, C exists.

Recall: $E \xrightarrow{e} X$ is called monic if $F \overset{h}{\underset{k}{\rightrightarrows}} E \xrightarrow{e} X$ $eh = ek \Rightarrow h = k$.

i.e. $e \circ (-)$ is injective $\mathrm{Hom}(F, E) \to \mathrm{Hom}(F, X)$ $\forall F$.

For Set, Vect, etc, monic $\Leftrightarrow$ injective.

Similarly, $Y \xrightarrow{c} C$ is epic if $(-) \circ c$ is injective from $\mathrm{Hom}(C, F) \to \mathrm{Hom}(Y, F)$

For sets, Vect, etc epic $\Leftrightarrow$ surjective

Exercise: equalizers are monic, coeq are epic.

~~[illegible]~~

Def: Suppose $\mathcal{C}$ has zero. A kernel $(\mathrm{Ker} f, \ker f)$ is the equalizer of $X \overset{f}{\underset{0}{\rightrightarrows}} Y$, and $(\mathrm{Coker} f, \mathrm{coker} f)$ is coequalizer.

(if it exists)

(No mention of elements!)

Easy: In additive cat, $\mathrm{Ker}(f-g) = \mathrm{Eq}(f, g)$ so kernels exist $\Rightarrow$ equalizers exist.

Note: Not every monic map is a kernel.

Ex: inclusion of non-normal subgp in Gps.

Stupid examples: $\mathrm{Ker} f$ does not exist in EvenDimVect.

Many expected properties hold, prove w/ univ. props not elements

Thm: A additive.

1. $M \cong 0 \iff id_M = 0 \iff End(M) = 0$
2. $M \xrightarrow{f} N$ monic $\iff 0 \cong \ker f$; epic then $0 \cong \operatorname{Coker} f$
3. $\ker(M \xrightarrow{0} N) \cong (M \xrightarrow{id} M) \cong \operatorname{Coker}(N \xrightarrow{0} M)$
4. Spse $\ker f$ exists.

$$\begin{tikzcd} & & L \arrow[dll, dashed, "\tilde{\pi}"'] \arrow[d, "\pi"] & & \\ \ker f \arrow[rr, "\ker f"] & & M \arrow[r, "f"] & N \arrow[r, "\operatorname{coker} f"] \arrow[d, "k"] & \operatorname{Coker} f \arrow[dl, dashed, "\tilde{k}"] \\ & & & P & \end{tikzcd}$$

$\exists \tilde{\pi} \iff f\pi = 0$, in which case $\tilde{\pi}$ is unique.

$\exists \tilde{k} \iff kf = 0$, ...

BUT: $M \xrightarrow{f} N$ both mono+epi does NOT imply f is isom!!!

Ex 1: $\mathcal{C}$ = homotopy cat of $\mathbb{Z}$-modules.

$$\begin{array}{rccc} M = & (\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}) \\ f\downarrow & 0\downarrow & & q\downarrow \\ N = & (0 & \to \mathbb{Z}/2 \to & 0) \end{array}$$

Claim: $0 \cong \operatorname{Coker} f \cong \ker f$

(In $Ch(\mathbb{Z})$, $\operatorname{Coker} f \cong 0$ and $\ker f = (\mathbb{Z} \xrightarrow{2} 2\mathbb{Z}) \cong (\mathbb{Z} \xrightarrow{1} \mathbb{Z})$

in $K(\mathbb{Z})$, $\ker f \simeq 0$)

But f has no inverse

Ex 2: Come up w/ stupid ad hoc cat. where it fails.

Def: An abelian cat is an additive cat where

1. $\ker f$, $\operatorname{Coker} f$ exist $\forall f$
2. Every monic is a kernel

   Every epic is a cokernel

<u>Prop</u>: In ab. cat, f mono + epi $\Rightarrow$ f isom.

<u>Pf</u>: f mono $\Rightarrow$ $f = \ker g$ for some g. $M \xrightarrow{f} N \underset{0}{\overset{g}{\rightrightarrows}} P$.

But f epi $\Rightarrow$ $g = 0$. $\mathrm{Ker}(0)$ is isom. $\boxed{\checkmark}$

<u>Exercise</u>: g mono $\Rightarrow$ $g = \ker(\mathrm{coker}\, g)$ <u>Cor</u>: Every kernel is kernel of an epi.

<u>The big nasty</u> $\mathcal{A}$ additive.

$$\begin{tikzcd}
\mathrm{Ker} f \arrow[r, hook, "\ker f"] & M \arrow[rr, "f"] \arrow[d] \arrow[drr, dashed, "\varphi"] & & N \arrow[r, two heads, "\mathrm{coker} f"] & \mathrm{Coker}\, f \\
& \mathrm{Coker}(\ker f) \arrow[urr, dashed, "\psi"] \arrow[rr, dashed] & & \mathrm{Ker}(\mathrm{coker} f) \arrow[u] &
\end{tikzcd}$$

"quot of M" $\quad \mathrm{Coker}(\ker f) \equiv \mathrm{Coim}(f) \overset{\iota}{\dashrightarrow} \mathrm{Im}(f) \equiv \mathrm{Ker}(\mathrm{coker} f) \quad$ "sub of N"

- φ, ψ exist by lifting (e.g. $\mathrm{coker} f \circ f = 0$ so φ exists)
- $\mathrm{coker} f \circ \psi = 0$, $\varphi \circ \ker f = 0$. (e.g. $\mathrm{coker} f \circ (f \circ \ker f) = 0$ so has unique descent, both 0 and $\varphi \circ \ker f$ are descents)
- so ι exists, descent of both φ and ψ. (descents unique)
- ι is both epi and mono. (yuck...) (WTS φ epi, ψ mono)

But ι need not be isom (see earlier example in $K(\mathbb{Z})$.)

i.e. $M/\ker f \not\cong \mathrm{Im} f$ in general.

But for abelian cat, ι is isom, $\mathrm{Coim}(f) \xrightarrow{\sim} \mathrm{Im}(f)$ canonically.

(equivalent defn of abelian)

Cor: In ab cat, $M \xrightarrow{f} N$ mono-epi decomp.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, two heads] & & N \\ & \mathrm{Im} f \arrow[ur, hook] & \end{tikzcd}$$

Def: In ab cat, a <u>subobject</u> of M is a monic map $S \hookrightarrow M$.

Two are <u>equivalent</u> if

$$\begin{tikzcd} S \arrow[dr, hook] \arrow[d, "\wr", leftrightarrow] & \\ S' \arrow[r, hook] & M \end{tikzcd}$$

No mention of elements

A <u>quotient</u> is ...

Cor: $\{\text{subobjects}\}/\sim \;\Longleftrightarrow\; \{\text{quotients}\}/\sim$

$(S \overset{s}{\hookrightarrow} M) \longmapsto \mathrm{Coker}\, s$

$(\mathrm{Ker}\, q) \longleftarrow\!\!\shortmid\; M \overset{q}{\twoheadrightarrow} Q$

Def: $A \xrightarrow{f} B \xrightarrow{g} C$ is <u>exact at B</u> if
1) $gf = 0$
2) $\mathrm{Ker}\, g \equiv \mathrm{Im} f$ as subobjects of B.

Note: Always $\mathrm{Im} f \hookrightarrow \mathrm{Ker}\, g$ by ① so can construct

$$\mathrm{Ker}\, g / \mathrm{Im} f \quad \left(\cong \mathrm{Ker}(\mathrm{Coim} f \to \mathrm{Coker}\, g)\right)$$

$\| \; \mathrm{Coker}(\mathrm{Im} f \to \mathrm{Ker}\, g)$

which measures failure of exactness.

Fine, but: HOW TO PROVE SNAKE LEMMA w/o elements!

<u>Approach 1</u>: Freyd ('64): $\mathcal{A}$ small ~~additive~~ abelian cat. $\exists$ additive

$F : \mathcal{A} \to R\text{-Mod}$ for some (crazy) ring R.

Mitchell ('65): F can be made <u>full</u> + faithful + exact.

full: if you construct δ in R-Mod you get it in $\mathcal{A}$.

NOT essentially surjective!
$\mathcal{A}$ need not have projectives...
like $\mathbb{Q}/\mathbb{Z}$-mod fd.

Ok, that's cheating.

Method 2: Use <u>members</u>, not elements.

<u>Def:</u> ($\mathcal{A} \in \mathcal{A}$ abelian cat.) A <u>premember</u> of A is a map $M \xrightarrow{x} A$ for some $M \in \mathcal{A}$

(Think: if $A \in R\text{-Mod}$, $a \in A$, have a map $\begin{array}{c} {}_R R \to A \\ 1 \mapsto a \end{array}$)

Two premembers are <u>equiv</u> ($x \equiv y$) if

$$\exists\, \begin{tikzcd} & M \arrow[dr, "x"] & \\ P \arrow[ur, two heads, "u"] \arrow[dr, two heads, "v"'] & & A \\ & N \arrow[ur, "y"'] & \end{tikzcd}$$

(and both are equiv to $x \circ u : P \to A$)

<u>Claim 1:</u> Equiv reln. A <u>member</u> is an equiv class

~~Note: $M \xrightarrow{x} A$ equiv to $M \oplus N \xrightarrow{p_M} M \to A$. So any two members have a common source.~~

<u>Note:</u> $M \xrightarrow{x} A$ equiv to

$$\begin{tikzcd} & M \arrow[dr, "x"] & \\ M \arrow[ur, two heads, "id"] \arrow[dr, two heads, "x"'] & & A \\ & \operatorname{Im} x \arrow[ur, hook] & \end{tikzcd}$$

(epi mono decomp)

So every member has a <u>unique</u> subobject. (up to equiv)

(Unique b/c $\operatorname{Im} x = \operatorname{Im} x \circ u$ since u, v epi.)
$= \operatorname{Im} y \circ v = \operatorname{Im} y$

Members are a more convenient way to discuss subobjects, b/c

1) Any two members can be given a common source.
$$(M \xrightarrow{x} A) \equiv (M \oplus N \xrightarrow{p_M} M \xrightarrow{x} A)$$

2) Members are functorial.
$$\begin{tikzcd} A \arrow[r, "f"] & B \\ M \arrow[u, "x"] \arrow[ur, dashed] & \end{tikzcd}$$
$f \circ (-)$ sends $\operatorname{Mem}(A) \to \operatorname{Mem}(B)$

3) f monic $\iff \forall x \in \operatorname{Mem}(A)$, $fx \equiv 0 \Rightarrow x \equiv 0$. (Not same as saying f injective on members!)
f epic $\iff \forall y \in \operatorname{Mem}(B)\ \exists x \in \operatorname{Mem}(A)$ w/ $f(x) \equiv y$.

$A \xrightarrow{f} B \xrightarrow{g} C$ exact $\iff$ $gf = 0$ and $\forall x \in \operatorname{Mem}(B)$ w/ $gx = 0$, $\exists y \in \operatorname{Mem}(A)$ w/ $f(y) = x$.

Now you can prove 5-lemma w/ Members rather than elements! (10)

Complication: Despite ①, you can't add members!! Can add premembers w/ same source but doesn't preserve equiv.

Ex: $\mathcal{A}$ = Vect $\quad A = \mathbb{C}^2 \quad$ Memb(A) = $\{0, \text{all lines in } \mathbb{C}^2, \mathbb{C}^2\}$ hmm.

Ex: $A = \mathbb{C}$

$\mathbb{C} \xrightarrow{1} \mathbb{C}$ is a member
$\mathbb{C} \xrightarrow{-1} \mathbb{C}$ is a member $\quad$ All equiv, $\quad 1+1=2$
$\mathbb{C} \xrightarrow{2} \mathbb{C}$ is a member. $\quad 1+(-1)=0$ ← NOT equiv

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Method 3: Members are great for checking properties of maps (inj, surj) but not for constructing maps. E.g. for Snake lemma can construct "Memb(δ)": Memb(Ker f_3) → Memb(Coker f_1), but we need this to come from δ.

A "better" proof of Snake lemma: use fiber squares.

$$\begin{tikzcd} X\times_Z Y \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

$X\times_Z Y$ is final object in cat
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$
pullback

Exists in abelian cat as follows:

$$\underbrace{\left(X\times_Z Y \rightarrow\right)}_{\text{equalizer}} X\oplus Y \underset{g\circ p_Y}{\overset{f\circ p_X}{\rightrightarrows}} Z$$

Properties:
- g surjective ⇒ $\tilde{g}$ surjective
- g injective ⇒ $\tilde{g}$ injective

Moreover, Ker $g \cong$ Ker $\tilde{g}$ naturally.

Similarly, have pushout
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X\times^Z_X Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$
via Coeq($Z \rightrightarrows X\oplus Y$)

$$\begin{tikzcd}
 & & & \delta & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Ker} f_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Coker} f_1 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & Q \arrow[r] & B_3 \arrow[r] & 0
\end{tikzcd}$$

Exact rows but not columns.

Let $\tilde{\delta} : P \to Q$ be $P \to A_2 \to B_2 \to Q$.

(Use member / diagram chase to) prove

$$\begin{tikzcd}
P \arrow[d, "\tilde{\delta}"'] & \\
Q \arrow[r] & B_3
\end{tikzcd} = 0 \qquad
\begin{tikzcd}
A_1 \arrow[r] & P \arrow[d] \\
 & Q
\end{tikzcd} = 0$$

Then descent shows that $\tilde{\delta}$ induces $\operatorname{Ker} f_3 \xrightarrow{\delta} \operatorname{Coker} f_1$

$$\operatorname{Ker} f_3 = \operatorname{Coker}(A_1 \to P), \qquad \operatorname{Coker} f_1 = \operatorname{Ker}(Q \to B_3)$$

We'll use pushouts / pullbacks many times. It's a trick for finding morphisms of S.E.S.

Given
$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, hook] \arrow[d, equal] & P \arrow[r, squiggly] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner"] & C_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & 0
\end{tikzcd}$$
want to fill it in.