What is hom. alg? By end of today, will hopefully answer.

- Toolbox in study of categories w/ alg. structure: modules, sheaves, etc. Packaged as additive, abelian cats.

Roughly, Additive: { • Hom spaces are $\mathbb{Z}$-modules (can add, $\exists$ zero map)
• $X \oplus Y$ exists, 0 exists }

Abelian: • Also, have kernels, cokernels, images for all morphisms.

Examples: (1) R-Mod is abelian. So are many subcategories:

(1a) R Noeth then R-mod is abelian. Notation: lower case = f.g.
(assume unless otherwise stated)

(1b) Subex: $\mathrm{Vect}_{\mathbb{k}}$, $\mathrm{vect}_{\mathbb{k}}$ abelian.

(1c) Reps of quivers (next talk). But these are secretly R-mod!

(1d) R graded. R-gmod. But this is secretly embedded in R_h-mod for some ring R_h (Gorav)

(1e) Ch(R-mod). These are: Objects: $M^{i-1} \xrightarrow{d_{i-1}} M^i \xrightarrow{d_i} M^{i+1}$ $d_i \circ d_{i-1} = 0$ i.e. "$d^2 = 0$"

Morphisms:
$$\begin{tikzcd} M^{i-1} \arrow[r] \arrow[d, "f^{i-1}"'] & M^i \arrow[r] \arrow[d, "f^i"'] & M^{i+1} \arrow[d, "f^{i+1}"'] \\ N^{i-1} \arrow[r] & N^i \arrow[r] & N^{i+1} \end{tikzcd}$$
s.t. squares commute
chain maps

Direct sum, kernels, are defined termwise: $\mathrm{Ker}(f^\bullet) = \; \to \mathrm{Ker} f^i \to \mathrm{Ker} f^{i+1} \to$

and this "works"

Again, secretly $\mathrm{Ch}^b(R\text{-mod}) \cong R'\text{-gmod}$ for some R'.
need boundedness, whatever.
to match lowercase.

$R' = R[d]/d^2$
$\deg d = 1$,
$\deg R = 0$

(2) Sheaves(X) is abelian, NOT $\cong$ R-mod for some ring!

(Morita theory: When $\mathcal{A} \cong R$-mod ?) (Hint: possibly in multiple ways!)

(3) Additive: $R\text{-proj} = $ direct summands of (f.g.) free modules

R-free ← seems more natural but is actually worse.

(3a) FinDimVect (no kernels, images) ← seems unnatural, b/c not closed under taking direct summands just like R-free!

Def: An additive category is Karoubian if it is closed under taking direct summands.

(This makes sense for subcats of R-Mod. $\exists$ general defn)

$\exists$ construction, Karoubi envelope, which formally adds all direct summands. B/c you can +shold, may as well assume all add. cats are Karoubian. $\mathrm{Kar}(R\text{-free}) \cong R\text{-proj}$

Hom alg is used to study the cats you care about. I care about R-mod! We'll do some baby rep theory to develop some fun examples. How to think about add + abelian.

Hom alg also has intrinsic interest, w/ research problems, we'll discuss in week 10.

So what does hom. alg. do to an abelian/add cat? Leave room above + to right.

$$\mathcal{A} \longrightarrow Ch(\mathcal{A}) \longrightarrow \underset{\text{homotopy category}}{K(\mathcal{A})}$$

$M \longmapsto (0 \to \underline{M} \to 0)$ (underline mean in deg 0.)

quotient of $Ch(\mathcal{A})$. Don't change objects.

$$\mathrm{Hom}_K(M^\bullet, N^\bullet) = \mathrm{Hom}_{Ch}(M^\bullet, N^\bullet) / \text{nullhomotopies}$$

(**Def:** Given a homotopy

$$\begin{tikzcd} M^{i-1} \arrow[r] & M^i \arrow[r] \arrow[dl, "h^i"'] & M^{i+1} \arrow[dl, "h^{i+1}"'] \\ N^{i-1} \arrow[r] & N^i \arrow[r] & N^{i+1} \end{tikzcd}$$

get a chain map $dh+hd$, called a nullhomotopy (nullhomotopic))

Now backwards (above): h^i $\quad h^i(M^\bullet) = \mathrm{Ker}\, d_i / \mathrm{Im}\, d_{i-1}$. Extends to $K(\mathcal{A})$ since

$h^i(f) = 0$ if f is nullhomotopy.

Rmk: Can make $Ch(\mathcal{P})$, $K(\mathcal{P})$ if $\mathcal{P}$ is additive. Can't do h^i, b/c no Ker, Im, Coker or the rest of the story.

Abelian cats were invented for this purpose!

Weird stuff in $K(\mathcal{A})$: Consider ($\mathcal{A}$ = $\mathbb{Z}$-mod)

$$\begin{tikzcd} f^\bullet \quad \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\ \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & {} \end{tikzcd}$$

Then: $\mathrm{Coker}\, f \cong 0$ $\quad$ $\mathrm{Ker}\, f \cong 0$ in $K(\mathcal{A})$! $\quad$ f induces isom on all h^i ← **Def:** quasi-isom

BUT $\nexists f^{-1}$ b/c no nonzero maps $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. A qiso, but NOT an iso.

$K(\mathcal{A})$ is NOT an abelian cat!

<u>Def:</u> $D(\mathcal{A})$, the <u>derived cat</u>, is obtained by formally inverting all qisos. (Still not abelian)

The point: replace crappy complexes (like $0 \to \mathbb{Z}/2\mathbb{Z} \to 0$) with nicer ones (all chain objects in $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ are free!)

Add arrow $K(\mathcal{A}) \xrightarrow{h^i} D(\mathcal{A})$. K and D are <u>triangulated</u> — have notions of "long exact sequence" instead of short exact sequence. "Cones NOT Cokernels"

<u>Shouldn't make sense yet...</u>

Hom alg studies all these setups, and functors b/w them! Key issue:

$$\begin{tikzcd} F: \mathcal{A} \arrow[r] & \mathcal{B} \\ F: Ch(\mathcal{A}) \arrow[u, "h^i"] \arrow[r] & Ch(\mathcal{B}) \arrow[u, "h^i"] \end{tikzcd}$$

Induces

abelian cats, F additive

THIS SQUARE DOES NOT COMMUTE

Measuring the failure to commute is our first step.

Really, this measurement occurs by looking at $F: D(\mathcal{A}) \to D(\mathcal{B})$ but we'll do more <u>elementary stuff first.</u>

<u>Aside:</u> Alg Top studies functors $Top \to Ab$. Topology has many structures <u>like</u> triangulated cats: cones, suspensions, etc, so these functors are similar to h^i. There is a rigorous dictionary using simplicial objects, Dold-Kan thm. Homotopical alg. Not this course.

<u>Outline:</u>

- Examples — reps of $\mathbb{C}[x]$, quiver reps.
- Additive + Abelian cats ←
- Hom alg standards (lots of it!)
- Groth groups, Morita theory
- Spectral sequences
- Hopfological algebra
- Koszul duality (if time)