What is hom. alg? By end of today, will hopefully answer.

- Toolbox in study of categories of alg structure: modules, sheaves, etc. Packaged as additive, abelian cats.

Roughly, **Additive:**
- Hom spaces are $\mathbb{Z}$-modules (can add, $\exists$ zero map)
- $X \otimes Y$ exists, $0$ exists

**Abelian:**
- Also, have kernels, cokernels, images for all morphisms.

**Examples:**

1. **$R$-Mod is abelian.** So are many subcategories:
   - **1a** $R$ Noether then $R$-mod is abelian, Notation: lower case $= f_{\mathfrak{g}}$.
   - **1b** Subext: Vect, vect is abelian.
   - **1c** Reps of quivers (next talk). But there are secretly $R$-mod!
   - **1d** $R$ graded. $R$-grad. But this is secretly embedded in $R_{\mathfrak{g}}$-mod for some ring $R_{\mathfrak{g}}$ (Grasse).

2. **Ch$(R$-mod$)$:** There are: Objects: $M^i \xrightarrow{d^i} M^{i+1} \xrightarrow{d^i} 0$ s.t. square commute.
   - Morpops: $(N \xrightarrow{f} M \xrightarrow{g})^i \xrightarrow{d^i} (N \xrightarrow{f} M^{i+1} \xrightarrow{g})$

   Direct sum, kernel, are defined tannoy: $\text{ker}(f^i) = \text{ker}(f_{\mathfrak{g}})$ and the "woks".

   Again, secretly $\text{Ch}(R$-mod$) \cong R'$-grad for some $R'$.

3. **Sheaves ($X$) is abelian, NOT $\cong R$-mod for some ring!**
   
   (Motto theory: When $A \cong R$-mod?),(that: possibly in multiple ways!)

3a. **Additive: $R$-proj $= \text{direct summands of} (f_{\mathfrak{g}})$ free modules.

   $R$-free $\rightarrow$ seem more natural, but $\not\cong$ actually worse.

3b. **Evn-DimVect (no kernels, images) $\not\cong$ seem unnatural, be not closed under take direct sum, $\not\cong R$-free!
Def: An additive category is Karoubian if it is closed under taking direct sums.  
(This makes sense for subcats of R-mod. I general don't.)

Example: Karoubi envelope, which formally adds all direct sums. But you can pretend they may as well assume all add cats are Karoubian. \( \text{Kar}(R\text{-free}) = R\text{-proj} \)

Hom alg is used to study the cats you care about. I care about R-mod! We'll do some baby rep theory to develop some fun examples. How to think about additivity.

Hom alg also has intrinsic interest, w/ research problems, we'll discuss in week 10.

So what does hom alg. do to an abelian add cat? Leave room above, to right.

\[
\begin{align*}
A & \longrightarrow \text{Ch}(A) \longrightarrow \text{K}(A) \\
M & \longrightarrow (0 \rightarrow M \rightarrow 0) \\
& \text{(underline mean in deg 0.)} \\
& \text{quotient of Ch}(A), \text{Don't change objects.}
\end{align*}
\]

\[
\text{Hom}_K(M^i, N^i) = \text{Hom}_{\text{Ch}}(M^i, N^i) / \text{null homotopies}
\]

(Def: Given a homotopy \( h^i : M^i \rightarrow M^{i+1} \), get a chain map dhithat, called a null homotopy \( (\text{null homotopic}) \).)

Now backwards (above): \( h^i(f) = 0 \) if \( f \) is null homotopy.

لاق: Can make \( \text{Ch}(P), K(P) \) if \( P \) is additive. Can't do \( h^i \) w/ blc no \text{Ker}, \text{Im}, \text{Coker}

Abelian cats were invented for this purpose!

Weird stuff in \( K(A) \): Consider \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \ldots \)

\[
\begin{align*}
\mathbb{Z}\text{-mod} & \rightarrow \downarrow \rightarrow \downarrow \rightarrow \downarrow \rightarrow \\
0 & \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \\
\text{Then} \ Coker(f) & = 0 \quad \text{Ker}(f) = 0 \in \text{K}(\mathbb{Z})! \quad \text{finds role on all} \ h^i \quad \text{Def: question}.
\end{align*}
\]
But if \( \mathfrak{A} \) is no nonzero rep, \( \mathbb{Z}/2 \to \mathbb{Z} \). A qism, but not m from \( \mathfrak{A} \).

\( K(\mathfrak{A}) \) is NOT an abelian cat!

\textbf{Def.} \( D(\mathfrak{A}) \), the derived cat, is obtained by formally inverting all qisms. (Still not abelian.)

The point: replace crappy complexes (like \( 0 \to \mathbb{Z}/2 \to \mathbb{Z} \)) with nice ones (all chain objects in \( \to \mathbb{Z} \to \mathbb{Z} \to 0 \) are free).

Add arrow \( K(\mathfrak{A}) \to D(\mathfrak{A}) \). \( K \) and \( D \) are \textit{translated} - have notions of "long exact sequence" instead of short exact sequence. "Core NOT Cokernel!"

Shouldn't make sense yet...

Hom alg studies all these setups, and factors blur then! Key issue:

\[ F : \mathfrak{A} \to \mathfrak{B} \text{ abelian cats, } F \text{ additive} \]

\[ F_i \] \text{ THIS SQUARE DOES NOT COMMUTE.} \text{ Induce } F : \mathfrak{C}(\mathfrak{A}) \to \mathfrak{C}(\mathfrak{B}) \text{ Measuring the failure to commute is own first step.}

Really, this measurement occurs by looking at \( F : D(\mathfrak{A}) \to D(\mathfrak{B}) \) but well do more -

elemenating stuff first.

\textbf{Aside:} Alg Top studies funtors \( \text{Top} \to \mathfrak{A} \). Topology has many structures like

\[ \text{translated cats: cores, suspens, etc, so these funtors are similar to } F_i \].

There is a rigorous dictionary using simplicial objects, Del-Ken thin. Homotopical alg.

Outline:

- Examples - reps of \( [\text{C}_\mathfrak{A}] \), qism reps.
- Additive + Abelian cats
- Hom alg standards (lots of it!) - Groth groups, North Surry
- Spectral sequence - Topological algebra - Knoted duality (future)