

What is hom. alg? By end of today, will hopefully answer.

• Toolbox in study of categories w/ alg. structure: modules, sheaves, etc. Packaged as additive, abelian cats.

Roughly, Additive: $\left\{ \begin{array}{l} \bullet \text{ Hom spaces are } \mathbb{Z}\text{-modules (can add, } \exists \text{ zero map)} \\ \bullet X \oplus Y \text{ exists, } 0 \text{ exists} \end{array} \right.$

Abelian: • Also, have kernels, cokernels, images for all morphisms.

Examples:

① $R\text{-Mod}$ is abelian. So are many subcategories:

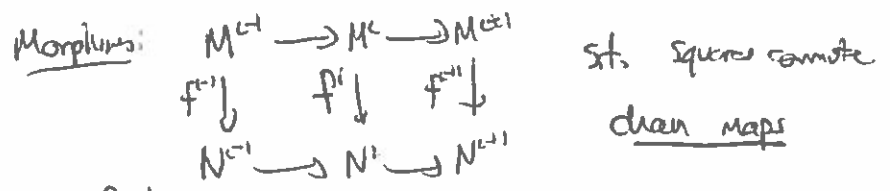
①a) R Noeth then $R\text{-mod}$ is abelian. Notation: lower case = f.g. (assume unless otherwise stated)

①b) Subex: $\text{Vect}_k, \text{vect}_k$ abelian.

①c) Repr of quivers (next talk). But these are secretly $R\text{-mod}$!

①d) R graded. $R\text{-gmod}$. But this is secretly embedded in $R_n\text{-mod}$ for some ring R_n (Exercise)

①e) $\text{Ch}(R\text{-mod})$. These are: Objects: $M^{(-1)} \xrightarrow{d_{-1}} M^0 \xrightarrow{d_0} M^{(1)}$ $d_i \circ d_{i-1} = 0$
i.e. " $d^2=0$ "



Direct sums, kernels, are defined termwise: $\text{Ker}(f^i) = \rightarrow \text{Ker}(f^i) \rightarrow \text{Ker}(f^{i+1}) \rightarrow$
 and this "works"

Again, secretly $\text{Ch}(R\text{-mod}) \cong R^1\text{-gmod}$ for some R^1 .
 (need boundedness, whatever, to match lowercase)

$P^1 = \mathbb{R}[x]/d^2$
 $\deg d = 1,$
 $\deg R = 0$

② Sheaves (X) is abelian, NOT $\cong R\text{-mod}$ for some ring!

(Morita theory: when $A \cong R\text{-mod}$?) (Hints possibly in multiple ways!)

③ Additive: $R^P\text{-proj} =$ direct summands of (f.g.) free modules

$R\text{-free}$ \leftarrow seems more natural but is actually worse.

③a) Even Dim Vect (no kernels, images) \leftarrow seems unnatural, b/c not closed under taking direct summands. In a lib. $P\text{-free}$!

Def: An additive category is Karoubian if it is closed under taking direct summands. (2)

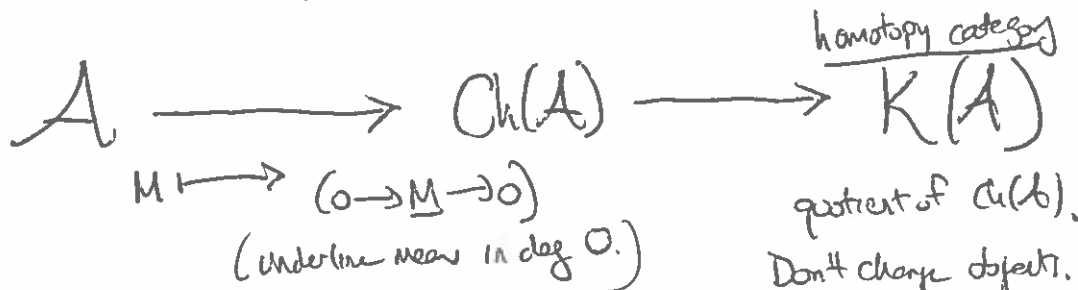
(This makes sense for subcats of $R\text{-Mod}$. \exists general defn)

\exists construction, Karoubi envelope, which formally adds all direct summands. Bk you can +shabd, may as well assume all add. cats are Karoubian. $\text{Kar}(R\text{-free}) \cong R\text{-proj}$

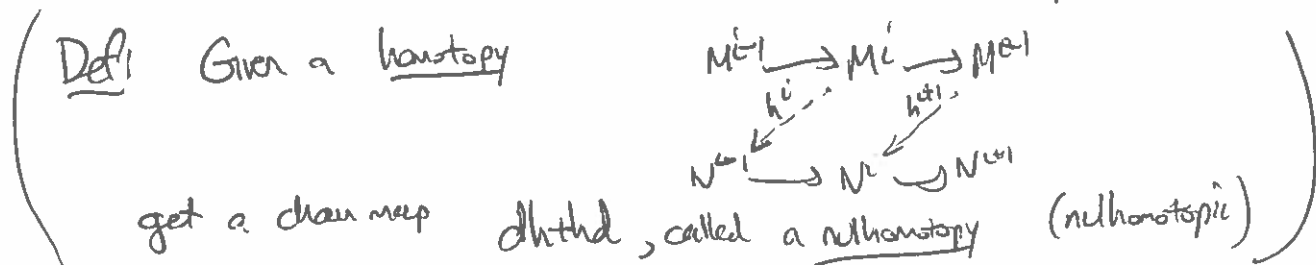
Hom alg is used to study the cats you care about. I care about $R\text{-mod}$! We'll do some baby rep theory to develop some fun examples. How to think about additabelian.

Hom alg also has intrinsic interest, w/ research papers, we'll discuss in week 10.

So what does hom. alg. do to an abelian/add cat? Learn room above + to right.



$$\text{Hom}_K(M^0, N^0) = \text{Hom}_{\text{Ch}}(M^0, N^0) / \text{nullhomotopies}$$



Now backwards (above): h^i $h^i(M^i) = \text{Ker } d^i / \text{Im } d^{i-1}$. Extends to $K(A)$ since

$h^i(f) = 0$ if f is nullhomotopy.

Rmk: Can make $\text{Ch}(P)$, $K(P)$ if P is additive. Can't do h^i b/c no Ker, Im, Coker or the rest of the story.
Abelian cats were invented for this purpose!

Weird stuff in $K(A)$: Consider $f: \mathbb{Z} \rightarrow \mathbb{Z}$ $\rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow 0 \rightarrow 0 \rightarrow \dots$

Then: $\text{Coker } f \cong 0$ $\text{Ker } f \cong 0$ in $K(A)$! f induces isom on all h^i Def: quasi-isom

BUT ∇F^{-1} b/c no nonzero maps $\mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}$. A qisom, but NOT an isom. ③

$K(A)$ is NOT an abelian cat!

Def: $D(A)$, the derived cat, is obtained by formally inverting all qisoms. (Still not abelian)

The point: replace crappy complexes (like $0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$) with nice ones (all chosen objects in $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$ are free)

Add arrow $K(A) \xrightarrow{h^i} D(A)$. K and D are triangulated - have notions of

"long exact sequence" instead of short exact sequence. "Cones NOT cokernels"
Shouldn't make sense yet...

Hom alg studies all these setups, and factors b/w them! Key issue:

$$F: A \rightarrow B \quad \text{abelian cats, } F \text{ additive}$$

$$\begin{array}{ccc} & h^i \uparrow & h^i \uparrow \\ \text{Invar} & F: Ch(A) & \rightarrow Ch(B) \end{array}$$

THIS SQUARE DOES NOT COMMUTE
 Measuring the failure to commute is our first step.

Really, this measurement occurs by looking at elementary stuff first.

$$F: D(A) \rightarrow D(B) \quad \text{but we'll do more}$$

Aside: Alg Top studies functor $Top \rightarrow Ab$. Topology has many structures like triangulated cats: cones, suspensions, etc, so these functors are similar to h^i .

There is a representations dictionary using simplicial objects, Dold-Kan thm. Homotopical alg. Not this course.

- Outline:
- Examples - reps of $[1, \infty]$ given reps.
 - Additive + Abelian cats
 - Hom alg standards (lots of $+$!)
 - Groth groups, Morita theory
 - Spectral sequences
 - Hopfological algebra
 - Koszul duality (if time)