Heading to Derived Functors

Coming Soon: $F: \mathcal{A} \to \mathcal{B}$ right exact.

$L^iF : \mathcal{A} \to \mathcal{B}$ defined by: for $M \in \mathcal{A}$ choose any $P^\bullet \twoheadrightarrow M$ proj res. Then $L^iF(M) := h^i(FP^\bullet)$

Big thm: s.e.s in $\mathcal{A}$ $\rightsquigarrow$ l.e.s. of L^iF.

More on projective resolutions

Suppose $P \overset{\varepsilon}{\twoheadrightarrow} A$, $Q \overset{\eta}{\twoheadrightarrow} C$ proj and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

Can we find R proj $\twoheadrightarrow B$ s.t.

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

?

Lemma 1 (Mini horseshoe):

1) Necessarily $R \cong P \oplus Q$ since top row splits.

2) If we can construct

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "\varepsilon"] & P \oplus Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "\eta"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

commuting then $P \oplus Q \twoheadrightarrow B$ by 5-lemma / snake lemma.

3) $P \oplus Q \xrightarrow[{[\varphi, \psi]}]{} B$ where $\varphi = f \circ \varepsilon$, ψ lifts η, so $g \circ \psi = \eta$.

Then squares commute ☑ (Note: ψ was not unique.)

Horseshoe lemma: Given $(P^\bullet) \twoheadrightarrow A$, $(Q^\bullet) \twoheadrightarrow C$ and $0 \to A \to B \to C \to 0$

$\exists\ (R^\bullet) \twoheadrightarrow B$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & P^\bullet \arrow[r] \arrow[d] & R^\bullet \arrow[r] \arrow[d] & Q^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

horseshoe b/c

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
P \arrow[r] & P \arrow[r] & P \arrow[r] & P \arrow[r] & A \arrow[r] \arrow[d] & 0 \\
 & & & & B \arrow[d] & \\
Q \arrow[r] & Q \arrow[r] & Q \arrow[r] & Q \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

as everyone who has shoed their horses knows...

Pf: By Lemma 1, can find

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker \varepsilon_A \arrow[r] \arrow[d] & P_0 \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \varepsilon_B \arrow[r] \arrow[d] & R_0 \arrow[r, "\varepsilon_B"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \varepsilon_C \arrow[r] \arrow[d] & Q_0 \arrow[r, "\varepsilon_C"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and by snake lemma, kernels form a s.e.s.!

Now we have

$$\begin{tikzcd}
 & 0 \arrow[d] \\
P_1 \arrow[r, two heads] & \ker \varepsilon_A \arrow[d] \\
 & \ker \varepsilon_B \arrow[d] \\
Q_1 \arrow[r] & \ker \varepsilon_C \arrow[d] \\
 & 0
\end{tikzcd}$$

so lemma 1 $\Rightarrow$ $\exists$

$$\begin{tikzcd}
R_1 \arrow[r, two heads] & \ker \varepsilon_B \arrow[d, hook] \\
 & R_0
\end{tikzcd}$$

Now repeat.

$\blacksquare$

What's great about an SES $0 \to P^\bullet \to Q^\bullet \to R^\bullet \to 0$ of complexes in $\mathrm{Proj}\,A$? Automatically termwise-split. (Not split though)
$\rightsquigarrow$ a Cone.

Now on the "functoriality" + non-functoriality of proj resolutions

Comparison Lemma:

$$\begin{tikzcd} P^\bullet \arrow[r, two heads] \arrow[d, dashed, "\exists \tilde f"'] & M \arrow[d, "f"] \\ Q^\bullet \arrow[r, two heads] & N \end{tikzcd}$$

$P^\bullet$ complex of projective (not nec. exact)

$Q^\bullet \to N$ exact (not nec. proj)

Then $\exists \tilde f$, and unique up to homotopy!

This is like a univ. property, so

Cor: Proj resolutions of M are unique up to homotopy equivalence, unique (up to homotopy)

Pf:

$$\begin{tikzcd} P^\bullet \arrow[loop left, "id"] \arrow[r] \arrow[d, shift right, "\exists"'] & M \arrow[d, equal] \\ (P')^\bullet \arrow[loop left, "id"] \arrow[r] \arrow[u, shift right, "\exists"'] & M \end{tikzcd}$$

Pf:

$$\begin{tikzcd} P^1 \arrow[r] \arrow[d, dashed, "f_1"'] & P^0 \arrow[r] \arrow[d, dashed, "f_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\ Q^1 \arrow[r] & Q^0 \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

- $\exists f_0$ by proj lifting $\begin{tikzcd} P_0 \arrow[dr] & \\ Q_0 \arrow[r, two heads] & N \arrow[r] & 0 \end{tikzcd}$
- $P^1 \to \downarrow \to$ is zero, so $\begin{tikzcd} P^1 \arrow[d] \\ Q^0 \end{tikzcd}$ lands in $\mathrm{Ker}\, d = \mathrm{Im}\, d$ $\begin{tikzcd} P^1 \arrow[dr] & \\ Q^1 \arrow[r, two heads] & \mathrm{Im}\, d \end{tikzcd}$

$\Rightarrow$ • $\exists f_1$ by proj lifting.

- etc.

So $\exists \tilde f$. If two lifts, difference is a lift of $M \xrightarrow{0} N$. WTS all lifts of $M \xrightarrow{0} N$ are nullhomotopic.

$$\begin{tikzcd} P^2 \arrow[r] \arrow[d, "f_2"'] & P^1 \arrow[r] \arrow[d, "f_1"'] \arrow[dl, "h_1"'] & P^0 \arrow[r, two heads] \arrow[d, "f_0"] \arrow[dl, dashed, "h_0"'] & M \arrow[r] \arrow[d, "0"] & 0 \\ Q^2 \arrow[r] & Q^1 \arrow[r] & Q^0 \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

- $P^0 \searrow = 0$ so $\begin{tikzcd} P^0 \arrow[d] \\ Q^1 \arrow[r, two heads] & \mathrm{Im}\, d \end{tikzcd}$ so $\begin{tikzcd} & P^0 \arrow[dl, dashed, "h_0"] \\ Q^1 & \end{tikzcd}$

$dh = f$

• $f_1 - h_0 d_P$ satisfies $d(f_1 - h_0 d_P) = f_0 d_P - f_0 d_P = 0$.

so $\exists$

$$\begin{tikzcd}
& P^1 \arrow[dl, "h_1"'] \arrow[d, "f_1 - h_0 d_P"] \\
Q^2 \arrow[r] & \operatorname{Im} d
\end{tikzcd}$$

i.e. $dh_1 + h_0 d = f_1$.

• etc. ∎

Does this make proj resolutions functorial? No!

$M \rightsquigarrow$ choose $P^\bullet$

$M \xrightarrow{f} N \rightsquigarrow$ choose $\tilde f$

composition preserved??

$$\begin{tikzcd}
P^\bullet \arrow[dr] \arrow[rrr, bend left, "\tilde f"] \arrow[ddd, bend right, "\tilde k"'] & & & Q^\bullet \arrow[dl] \arrow[dddlll, bend left, "\tilde g"] \\
& A \arrow[r, "f"] \arrow[d, "k = g\circ f"'] & B \arrow[dl, "g"] & \\
& C & & \\
R^\bullet \arrow[ur] & & &
\end{tikzcd}$$

Both $\tilde k$ and $\tilde g \circ \tilde f$ are lifts of $k = g\circ f$, so $\tilde k \simeq \tilde g\circ \tilde f$

But not necessarily equal.

Why not choose $\tilde k := \tilde g \circ \tilde f$?

Can't do it consistently.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, "k"] & B \arrow[d] \\
B' \arrow[r] & C
\end{tikzcd}$$

is lift of k going to be $\tilde g \circ \tilde f$ or $\tilde g' \circ \tilde f'$?

Analogous problem in topology — composition in π_1 is only assoc up to homotopy.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend left] & B \arrow[r, "g"] \arrow[rr, bend right] & C \arrow[r, "h"] & D
\end{tikzcd}$$

lifts association up to homotopy

$\tilde h \circ (\widetilde{gf}) \simeq (\widetilde{hg}) \circ \tilde f$.

To summarize: there is no functor $\mathcal{A} \to Ch^-(\mathrm{Proj}\,\mathcal{A})$

$M \longmapsto P^\bullet$

If $\exists$ proj's "enough" there is a functor $\mathcal{A} \to K^-(\mathrm{Proj}\,\mathcal{A})$.

$M \mapsto P^\bullet$

Recall: $F : \mathcal{A} \to \mathcal{B}$ is left exact if

$$0 \to A \to B \to C \rightsquigarrow 0 \to FA \to FB \to FC.$$

Ex: $\mathrm{Hom}(M,-)$ is left exact.

For bimodules, $(-)\otimes M$ is right exact.

we'll do some fun ones w/ quiver repr soon.

Prop: Left adjoints are right exact, and vice versa.

What about contravariant like $\mathrm{Hom}(-,M)$? Later!

FUND THM

Def/Thm: Let $F : \mathcal{A} \to \mathcal{B}$ be additive + right exact. $\mathcal{A}$ has enough projectives.

For $M \in \mathcal{A}$ pick $(P^\bullet) \twoheadrightarrow M$ and let $L^iF(M) := h^i(FP^\bullet)$

Then • $L^iF(M)$ is well defined (canonically indep of choice of $P^\bullet$)

• L^iF is a functor (functorial)

(*) • $L^0F \cong F$ canonically. $L^iF = 0$ for $i>0$.

(**) • If $0 \to A \to B \to C \to 0$ then get l.e.s in $\mathcal{B}$:

$$\cdots \to L^2F(C) \xrightarrow{\delta} L^1F(A) \to L^1F(B) \to L^1F(C) \xrightarrow{\delta} FA \to FB \to FC \to 0.$$

(***) Functorial in s.e.s.

• And L^iF is final among all functors satisfying (*) and (**) and (***).

Similarly, if F is left exact, $R^iF(M) := h^i(FI^\bullet)$ for inj resolution. ⑥

<u>Ex:</u> $\mathcal{A} = \mathbb{C}[x]/(x^4)\text{-mod}$ $\qquad \mathcal{B} = \mathrm{Vect}_{\mathbb{C}}$

$$F(M) = \{m \in M \mid xm = 0\} \quad \cdot F \cong \mathrm{Hom}\left(\mathbb{C}[x]/(x), -\right)$$

Inj res. of $M_{(0)}$ is $M_{(0)} \xrightarrow{x^3} \underbrace{\left(\underline{R} \xrightarrow{x} R \xrightarrow{x^3} R \xrightarrow{x} \cdots\right)}_{= I^\bullet}$

$F(R) = \overset{\mathbb{C}}{\cancel{}} = \mathbb{C}\cdot x^3$ so $FI^\bullet = \underline{\mathbb{C}} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \to \cdots$

$$R^iF(M_{(0)}) = \begin{cases} \mathbb{C} & \forall i \geq 0 \\ 0 & i < 0 \end{cases}$$

Same for $R^iF(M_{(0 b)})$ and $R^iF(M_{(0 b_b)})$ but $R^iF(M_{(0 b_{b_b})}) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i\neq 0 \end{cases}$

$$0 \to M_{(0)} \xrightarrow{x^3} \overset{R}{\overset{\shortparallel}{M}}_{(0 b_{b_b})} \to \overset{R/(x^3)}{\overset{\shortparallel}{M}}_{(0 b_b)} \to 0 \qquad \text{gives}$$

$$1 \longmapsto 1$$

$$0 \to \mathbb{C} \xrightarrow{\sim} \mathbb{C}\cdot x^3 \xrightarrow{0} \mathbb{C}\cdot x^2 \xrightarrow[\delta]{\sim} \mathbb{C} \xrightarrow{0} 0 \xrightarrow{\cancel{0}} \mathbb{C} \xrightarrow[\delta]{\sim} \mathbb{C} \xrightarrow{0} 0 \to \cdots$$

$$0 \to M_{(0)} \to M_{(0 b)} \to M_{(0)} \to 0 \qquad \text{gives}$$

$$0 \to \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \to \cdots$$

why?

$R/(x) = M_{(0)} \overset{x}{\hookrightarrow} M_{\left(\begin{smallmatrix}0 & \\ & 0\end{smallmatrix}\right)} = R/(x^2)$ induces

$$\begin{tikzcd}
M_{(0)} \arrow[r, hook, "x"] \arrow[d, "x^3"'] & M_{\left(\begin{smallmatrix}0 & \\ & 0\end{smallmatrix}\right)} \arrow[d, "x^2"] \\
R \arrow[r, "1"] \arrow[d, "x^{\bullet}"'] & R \arrow[d, "x^2"] \\
R \arrow[r, "x"] \arrow[d, "x^3"'] & R \arrow[d, "x^2"] \\
R \arrow[r, "1"] \arrow[d, "x"'] & R \arrow[d, "x^2"] \\
R \arrow[r, "x"] & R
\end{tikzcd}$$

apply F

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "1"] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "0"] \\
\mathbb{C} \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "0"] \\
\mathbb{C} \arrow[r, "1"] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "0"] \\
\mathbb{C} \arrow[r, "0"] & \mathbb{C} \\
& \vdots
\end{tikzcd}$$

Proof of big thm uses all our tricks. I'll do proj version. F right exact.

i) $P^{-1} \to P^0 \to M \to 0 \rightsquigarrow FP^{-1} \to FP^0 \to FM \to 0$

so $\underset{\substack{\| \\ L^0F(M)}}{h^0}\left(\cdots \to FP^{-1} \to \underline{FP^0} \to 0\right)$ is FM.

2) $$\begin{tikzcd}
P^0 \arrow[dr] \arrow[d, "\exists f"'] & \\
(P')^0 \arrow[r] & M
\end{tikzcd}$$ then $$\begin{tikzcd}
FP^0 \arrow[d, "Ff"] \\
F(P')^0
\end{tikzcd}$$ induces $$\begin{tikzcd}
h^iFP^0 \arrow[d] \\
h^iF(P')
\end{tikzcd}$$.

f is unique up to homotopy. If $f \sim g$ then $\exists h$ s.t. $d_{P'}h + hd_P = f - g$.

Then $d_{FP'}F(h) + F(h)d_{FP} = Ff - Fg$. so $Ff \simeq Fg$.

Homotopic maps induce same map on homology.

So far we used: F preserves homotopies (b/c preserves addition)

F preserves isoms

3)

$$\begin{tikzcd} M \arrow[r] & N \\ P^\circ \arrow[u] \arrow[r, dashed, "\exists f"'] & Q^\circ \arrow[u] \end{tikzcd}$$

then $P^\circ \xrightarrow{f} Q^\circ$ induces $FP^\circ \xrightarrow{Ff} FQ^\circ$, unique up to homotopy.

4)

$$\begin{tikzcd} 0 \arrow[r] & P^\circ \arrow[r] \arrow[d] & R^\circ \arrow[r] \arrow[d] & Q^\circ \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Given $(0 \to A \to B \to C \to 0)$

- choose P°, Q°
- by horseshoe $\exists\, R^\circ$

Now FP° is NOT a complex of projectives, NOT exact (columns)

but $0 \to FP^\circ \to FR^\circ \to FQ^\circ \to 0$ is s.e.s. of complexes! (rows)

Why? F preserves $\oplus$ so it preserves split ses in $\mathcal{A}$.

H preserves termwise-splitness of ses of cxs

FR° is still a cone

The ses $0 \to FP \to FR \to FQ \to 0$ induces the desired l.e.s. $\blacksquare$

5) <u>Functoriality</u>: It's easy, is it??

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

induces map

twisted differential

$$\begin{tikzcd} 0 \arrow[r] & P^{\bullet} \arrow[r] \arrow[d] & (\widetilde{P^{\bullet} \oplus Q^{\bullet}}) \arrow[r] \arrow[d] & Q^{\bullet} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (P)' \arrow[r] & (\widetilde{P \oplus Q})' \arrow[r] & Q' \arrow[r] & 0 \end{tikzcd}$$

but squares need not commute!! $P^{\bullet} \to P' \oplus Q' \simeq P \to \ldots$ because both

lift $A \to B'$ = $A \to B'$, but not <u>equal</u>. For a morphism of SES need

genuine commutativity.

Need new result: a "homotopy morphism" of SES of complexes still induces

a map b/w LES.

This is, again, the non-functoriality of cones in disguise!

The book spends 2 pages fixing this + You MUST Read. Get used to this!

6) <u>Universality</u>: Nothing groundbreaking here, uses one key trick I'll

discuss later. (dimensional reduction)

Ext

Ext 1: $F = \mathrm{Hom}(M, -)$ is covariant, left exact. (Exact $\Leftrightarrow$ M projective)

$R^i F(N)$ denoted $\mathrm{Ext}^i_{(R)}(M,N)$

Compute as homology of complex $\mathrm{Hom}(M, I^\circ)$, i.e.

$$\mathrm{Hom}(M, I^0) \to \mathrm{Hom}(M, I^1) \to \mathrm{Hom}(M, I^2) \to \cdots$$

Ext 2: $G = \mathrm{Hom}(-, N)$ is contravariant left exact, meaning it is left exact as a functor $\mathcal{A}^{op} \to \mathcal{B}$.

$$0 \to A \to B \to C \rightsquigarrow \mathrm{Hom}(A,N) \leftarrow \mathrm{Hom}(B,N) \leftarrow \mathrm{Hom}(C,N) \leftarrow 0$$

To compute $R^i G(M)$, take injective resolution in $\mathcal{A}^{op}$, i.e.

projective resolution in $\mathcal{A}$!!! $\quad P^\circ \twoheadrightarrow M$

Homology of

$$\cdots \leftarrow \mathrm{Hom}(P^2, N) \leftarrow \mathrm{Hom}(P^1, N) \leftarrow \mathrm{Hom}(P^0, N) \leftarrow 0$$

Call this $\mathrm{Ext}^i_{(L)}(M,N)$

Thm (later): $\mathrm{Ext}^i_R(M,N) \underset{\text{canon}}{\cong} \mathrm{Ext}^i_L(M,N)$,

so compute however is easiest — Hom from proj or to inj.