

Heading to Derived Functors

①

Coming soon: $F: A \rightarrow B$ right exact.

$L^i F: A \rightarrow B$ defined by: for MSA choose any $P^\bullet \rightarrow M$

proj res Then $L^i F(M) := h^i(FP^\bullet)$

Big thm: ses in $A \rightsquigarrow$ l.e.s. of $L^i F$.

More on projective resolutions

Suppose

$$P \xrightarrow{\varepsilon} A$$

and

$$0 \rightarrow A \xrightarrow{f} B \rightarrow C \rightarrow 0$$

$$Q \xrightarrow{\eta} C$$

proj

Can we find $R \xrightarrow{\text{proj}} B$ s.t.

$$\begin{array}{ccccccc} 0 & \rightarrow & P & \rightarrow & R & \rightarrow & Q & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 & & 0 \end{array} \quad ?$$

Lemma 1 (Mittlerstufen):

1) Necessarily $R \cong P \oplus Q$ since top row splits.

2) If we can construct

$$\begin{array}{ccccccc} 0 & \rightarrow & P & \rightarrow & P \oplus Q & \rightarrow & Q & \rightarrow & 0 \\ & & \downarrow \varepsilon & & \downarrow & & \downarrow \eta & & \downarrow \\ 0 & \rightarrow & A & \xrightarrow{f} & B & \rightarrow & C & \rightarrow & 0 \end{array}$$

commute then

$P \oplus Q \rightarrow B$ by 5-lemma / snake lemma.

3)

$$P \oplus Q \xrightarrow{(\varphi, \psi)} B$$

where

$$\varphi = f \circ \varepsilon$$

ψ lifts η , so $g \circ \psi = \eta$.

then squares commute

(Note: ψ was not unique.)

Horseshoe lemma:

Given $(P^\bullet) \twoheadrightarrow A$ and $(Q^\bullet) \twoheadrightarrow C$ and $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ (2)

$\exists (R^\bullet) \twoheadrightarrow B$ s.t.

$$\begin{array}{ccccccc} 0 & \rightarrow & P^\bullet & \rightarrow & R^\bullet & \rightarrow & Q^\bullet \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0 \end{array}$$

horseshoe bc

$P \rightarrow P \rightarrow P \rightarrow P \rightarrow A \rightarrow 0$



$Q \rightarrow Q \rightarrow Q \rightarrow Q \rightarrow C \rightarrow 0$

as everyone who has shared their horses knows...

Pf: By Lemma 1,

can find $0 \rightarrow \text{Ker } \epsilon_A \rightarrow P_0 \xrightarrow{\epsilon_A} A \rightarrow 0$
 $0 \rightarrow \text{Ker } \epsilon_B \rightarrow R_0 \xrightarrow{\epsilon_B} B \rightarrow 0$
 $0 \rightarrow \text{Ker } \epsilon_C \rightarrow Q_0 \xrightarrow{\epsilon_C} C \rightarrow 0$

and by snake lemma, ~~kernel~~ kernel form a s.e.s.!

Now we have

$P_1 \twoheadrightarrow \text{Ker } \epsilon_A \rightarrow \text{Ker } \epsilon_B \rightarrow \text{Ker } \epsilon_C \rightarrow 0$

so $\text{Lemma 1} \Leftrightarrow \exists R_1 \twoheadrightarrow \text{Ker } \epsilon_B \rightarrow R_0$

Now repeat.



What's great about a SES $0 \rightarrow P^0 \rightarrow Q^0 \rightarrow R^0 \rightarrow 0$ of complexes is ③
Proj A? Automatically termwise-split. (Not split though)
 \rightarrow a Cone.

Now on the "functionality" + non-functionality of proj resolution

Comparison Lemma:

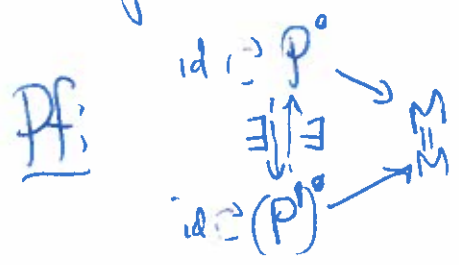
$$\begin{array}{ccc} P^0 & \twoheadrightarrow & M \\ \exists \tilde{f} \downarrow & & \downarrow f \\ Q^0 & \twoheadrightarrow & N \end{array}$$

P^0 complex of projectives (not nec. exact)
 $Q^0 \twoheadrightarrow N$ exact (not nec. proj)

Then $\exists \tilde{f}$, and unique up to homotopy!

This is like a univ. property, so

Cor: Proj resolutions of M are unique up to homotopy equivalence.
 Unique (up to homotopy)

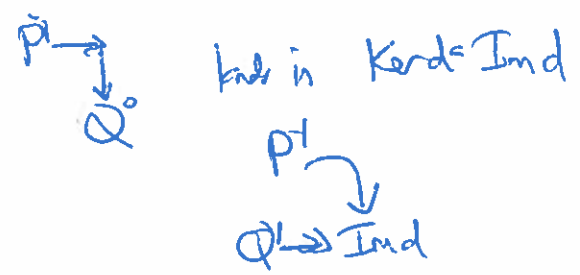


PF:

$$\begin{array}{ccccccc} P^1 & \rightarrow & P^0 & \rightarrow & M & \rightarrow & 0 \\ f_1 \downarrow & & f_0 \downarrow & & \downarrow h & & \\ Q^1 & \rightarrow & Q^0 & \rightarrow & N & \rightarrow & 0 \end{array}$$

$\exists f_0$ by proj lifting $\begin{array}{ccc} P_0 & \searrow & N \\ Q_0 & \twoheadrightarrow & N \end{array}$

$P^1 \rightarrow$ is zero, so



So $\exists \tilde{f}$. If two lifts, difference is a lift of $M \rightarrow N$. WTS all lifts of $M \rightarrow N$ are nullhomotopic.

$$\begin{array}{ccccccc} P^2 & \rightarrow & P^1 & \rightarrow & P^0 & \rightarrow & M \rightarrow 0 \\ \exists \tilde{f} \downarrow & \swarrow h & f_1 \downarrow & \swarrow h_0 & f_0 \downarrow & & \downarrow \\ Q^2 & \rightarrow & Q^1 & \rightarrow & Q^0 & \rightarrow & N \rightarrow 0 \end{array}$$

$\Rightarrow \exists f_1$ by proj lifting.
 etc

$P^0 = 0$ so $\begin{array}{ccc} P^0 & \rightarrow & \\ \downarrow & & \\ Q^1 & \rightarrow & \text{Im } d \end{array}$ so $\begin{array}{ccc} P^0 & \rightarrow & \\ \downarrow h_0 & & \\ Q^1 & \rightarrow & \text{Im } d \end{array}$
 $dh = f$

• $f_1 - h \circ p$ satisfies $d(f_1 - h \circ p) = f_{0,p} - f_{0,p} = 0$.

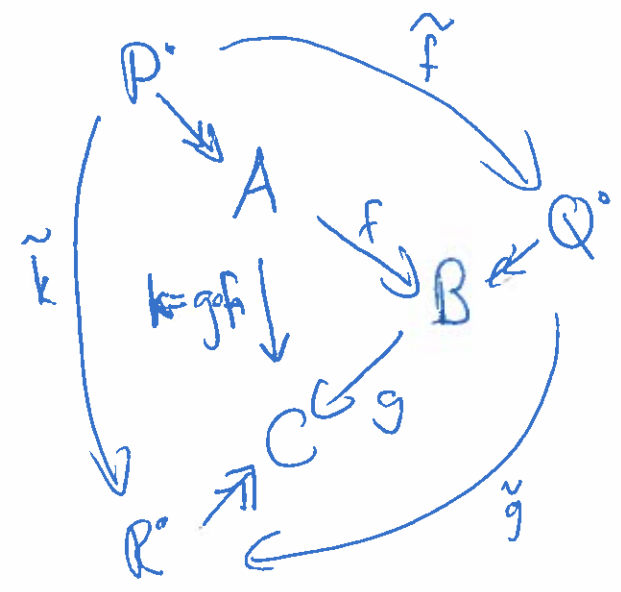
so \exists $\begin{matrix} p^1 \\ \swarrow h_1 \\ \mathbb{Q}^2 \end{matrix} \rightarrow \text{Im } d \begin{matrix} \downarrow f_1 - h \circ p \\ \end{matrix}$ is $dh_1 + h \circ d = f_1$.

• etc.

Does this make proj resolution functorial? No!

$M \rightsquigarrow \text{char } p^0$
 $M \xrightarrow{f} N \rightsquigarrow \text{char } \tilde{f}$

compatibility preserved??

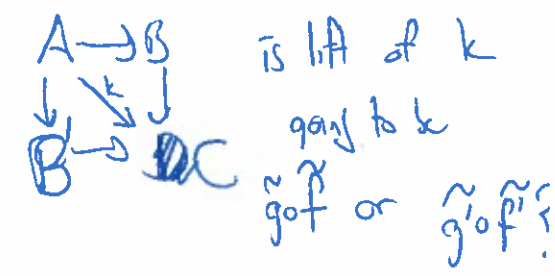


Both \tilde{k} and $\tilde{g} \circ \tilde{f}$ are lifts of $k = g \circ f$, so $\tilde{k} \simeq \tilde{g} \circ \tilde{f}$

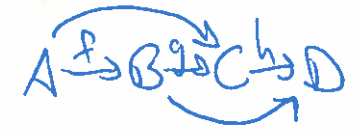
But not necessarily equal.

Why not choose $\tilde{k} := \tilde{g} \circ \tilde{f}$?

Can't do it consistently.



Analogous problem in topology — compatibility in Π_1 is only assoc up to homotopy.



lifts associative up to homotopy

$$\tilde{h} \circ (\tilde{a} \tilde{f}) \simeq (\tilde{h} \tilde{a}) \circ \tilde{f}$$

To summarize: there is no functor $A \rightarrow \text{Ch}(\text{Proj } A)$
 $M \mapsto P^\bullet$

If \exists Proj "enough" there is a functor $A \rightarrow K(\text{Proj } A)$. $M \mapsto P^\bullet$

Recall: $F: A \rightarrow B$ is left exact if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightsquigarrow 0 \rightarrow FA \rightarrow FB \rightarrow FC.$$

Ex: $\text{Hom}(M, -)$ is left exact.

For bimodules, $(-) \otimes M$ is right exact.

we'll do some fun ones w/ quiver repr soon.

Prop: Left adjoints are right exact, and vice versa.

What about contravariant like $\text{Hom}(-, M)$?
Later!

FWO THM

Def/Thm: Let $F: A \rightarrow B$ be right exact ^{additive +} A has enough projectives.

For $M \in A$ pick $(P^i) \twoheadrightarrow M$ and let $L^i F(M) := h^i(FP^i)$

Then $\bullet L^i F(M)$ is well defined (canonically indep of choice of P^\bullet)

$\bullet L^i F$ is a functor (functorial)

$\bullet L^0 F \cong F$ canonically. $L^i F = 0$ for $i > 0$.

\bullet If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ then get l.e.s in B :

$$\dots \rightarrow L^i F(C) \xrightarrow{\delta} L^i F(A) \rightarrow L^i F(B) \rightarrow L^i F(C) \xrightarrow{\delta} FA \rightarrow FB \rightarrow FC \rightarrow 0.$$

\bullet And $L^i F$ is final among all functors satisfying \bullet and \bullet and \bullet .

Functor in Ser.

Similarly, if F is left exact, $R^i F(M) := h^i(FI^\bullet)$ for inj resolution. ⑥

Ex: $A = \mathbb{C}[x]/(x^4)\text{-mod}$ $B = \text{Vect}_{\mathbb{C}}$

$F(M) = \{m \in M \mid xm=0\}$ $F \cong \text{Hom}(\mathbb{C}[x]/(x), -)$

Inj res. of $M_{(x)}$ is $M_{(x)} \xrightarrow{x^3} \left(\mathbb{R} \xrightarrow{x} \mathbb{R} \xrightarrow{x^3} \mathbb{R} \xrightarrow{x} \dots \right) = I^\bullet$

$F(\mathbb{R}) = \mathbb{C} = \mathbb{C} \cdot x^3$ so $FI^\bullet = \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \rightarrow \dots$

$R^i F(M_{(x)}) = \begin{cases} \mathbb{C} & \forall i \geq 0 \\ 0 & i < 0 \end{cases}$

Same for $R^i F(M_{(x^2)})$ at $R^i F(M_{(x^3)})$ but $R^i F(M_{(x^3)}) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i \neq 0 \end{cases}$

$0 \rightarrow M_{(x)} \xrightarrow{x^3} M_{(x)}^{\mathbb{R}} \rightarrow M_{(x^2)}^{\mathbb{R}/(x^2)} \rightarrow 0$ gives
 $1 \mapsto 1$

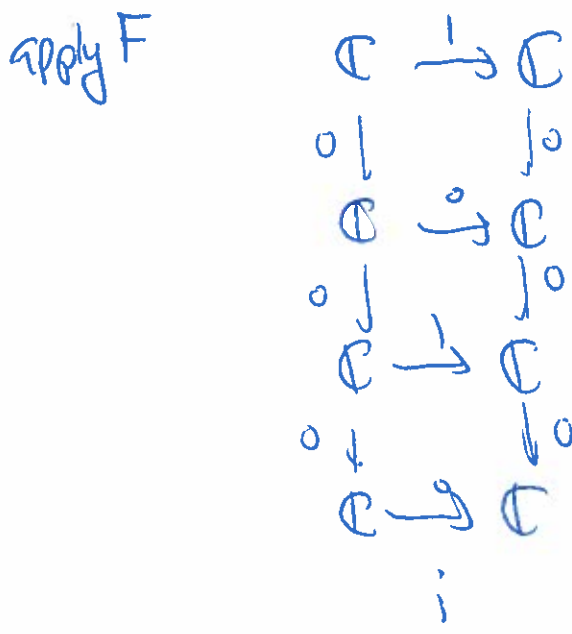
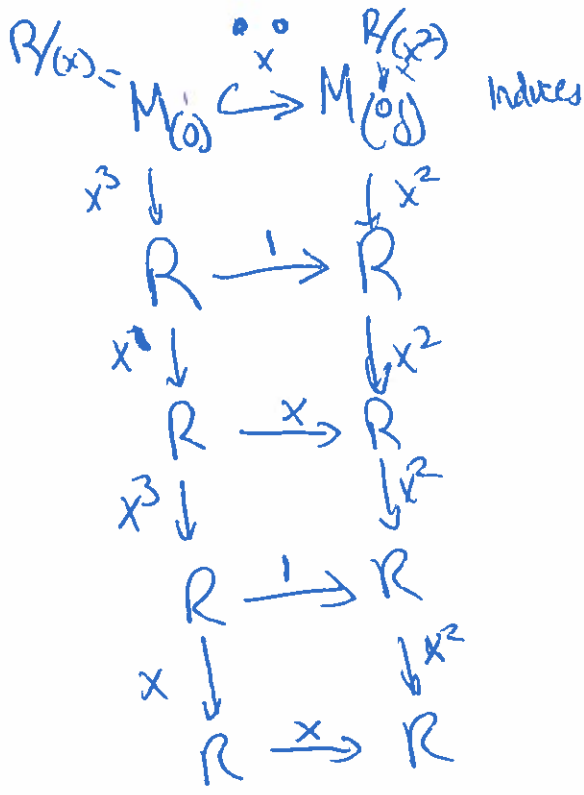
$0 \rightarrow \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{0} \mathbb{C} \rightarrow \dots$



$0 \rightarrow M_{(x)} \rightarrow M_{(x^2)} \rightarrow M_{(x)} \rightarrow 0$ gives

$0 \rightarrow \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \xrightarrow{0} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \rightarrow \dots$

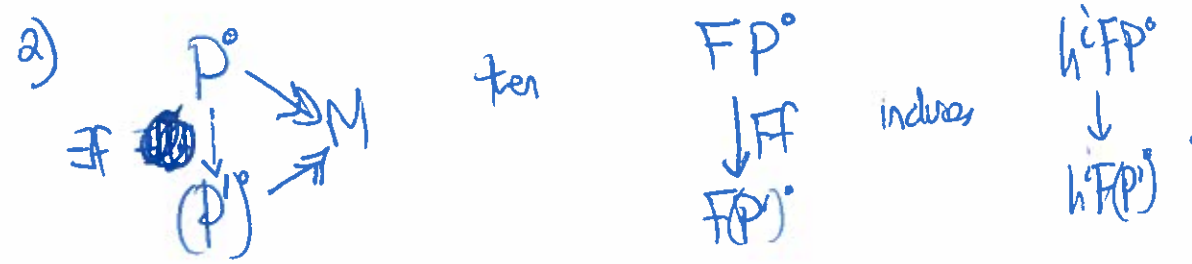
why?



Proof of big thm uses all our tricks. I'll do proj res. F right exact.

1) $P^1 \rightarrow P^0 \rightarrow M \rightarrow 0 \rightsquigarrow FP^1 \rightarrow FP^0 \rightarrow FM \rightarrow 0$

so $h^0(\dots \rightarrow FP^1 \rightarrow FP^0 \rightarrow 0)$ is FM .
 $L^0 F(M)$



f is unique up to homotopy. If f and g then $\exists h$ st. $dh + hd = f - g$.

Then $d_{FP} F(h) + F(h)d_{FP} = Ff - Fg$ so $Ff \approx Fg$.

Homotopic maps induce same map on homology.

So far we used:

F preserves homotopies (bk previous addtn) $\textcircled{8}$
 F preserves isms

3)
$$\begin{array}{ccc} M & \rightarrow & N \\ \uparrow & & \uparrow \\ P^\circ & \xrightarrow{f} & Q^\circ \\ \exists f & & \end{array}$$
 then $P^\circ \xrightarrow{f} Q^\circ$ induces $FP^\circ \xrightarrow{Ff} FQ^\circ$,
 unique up to homotopy.

4)
$$\begin{array}{ccccccc} 0 & \rightarrow & P^\circ & \rightarrow & R^\circ & \rightarrow & Q^\circ \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \text{Given } (0 & \rightarrow & A & \rightarrow & B & \rightarrow & C \rightarrow 0) \end{array}$$

- char P°, Q°
- by horseshoe $\exists R^\circ$

Now FP° is NOT a complex of projectives, NOT exact (columns)

but $0 \rightarrow FP^\circ \rightarrow FR^\circ \rightarrow FQ^\circ \rightarrow 0$ is seq of complexes! (rows)

Why? F preserves \oplus so it preserves ~~split~~ split seqs w/ A.

H preserves tensor-splitness of seqs of cxs

FR° is still a cplx

The seq $0 \rightarrow FP^\circ \rightarrow FR^\circ \rightarrow FQ^\circ \rightarrow 0$ induces the desired l.s. \square

5) Functoriality: It's easy, is it??

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

$$\downarrow \varphi \downarrow \psi \downarrow$$

$$0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0$$

induces map

↑ twisted differential

$$0 \rightarrow P' \rightarrow (P' \oplus Q) \rightarrow Q' \rightarrow 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \rightarrow (P') \rightarrow (P' \oplus Q') \rightarrow Q' \rightarrow 0$$

9

but squares need not commute!! $P' \rightarrow P' \oplus Q \rightarrow P' \oplus Q'$ because both

iff $A \hookrightarrow B = A \hookrightarrow B'$ but not equal. For a morphism of SES need genuine commutativity.

Need new result: a "homotopy morphism" of SES of complexes still induces a map btw lies.

This is, again, the non-functoriality of cones in disguise!

The book spends 2 pages fixing this + You MUST Read. Get used to this!

6) Universality: Nothing groundbreaking here, uses one key trick I'll discuss later. (dimension reduction)

Ext

Ext 1: $F = \text{Hom}(M, -)$ is covariant, left exact. (Exact $\Leftrightarrow M$ projective)

$R^i F(N)$ denoted $\text{Ext}_R^i(M, N)$

compute as homology of complex $\text{Hom}(M, I^\bullet)$, i.e.

$\text{Hom}(M, I^0) \rightarrow \text{Hom}(M, I^1) \rightarrow \text{Hom}(M, I^2) \rightarrow \dots$

Ext 2: $G = \text{Hom}(-, N)$ is contravariant left exact, meaning it is left exact as a functor $A^{\text{op}} \rightarrow B$.

$0 \rightarrow A \rightarrow B \rightarrow C \rightsquigarrow \text{Hom}(A, N) \leftarrow \text{Hom}(B, N) \leftarrow \text{Hom}(C, N) \leftarrow 0$

To compute $R^i G(M)$, take injective resolution in A^{op} , i.e.

projective resolution in A \parallel $P^\bullet \rightarrow M$

Homology of $\dots \leftarrow \text{Hom}(P^2, N) \leftarrow \text{Hom}(P^1, N) \leftarrow \text{Hom}(P^0, N) \leftarrow 0$

Call this $\text{Ext}_R^i(M, N)$

Thm (later): $\text{Ext}_R^i(M, N) \cong \text{Ext}_L^i(M, N)$

\Rightarrow compute homom is easiest \rightarrow Hom from proj or to inj.