Gaussian Elimination

Def: A complex is *exact* if $h^i(C^\cdot)=0$ $\forall i$.

A complex $C^\cdot$ is *contractible* if $id_C \simeq 0$, i.e. $C \underset{h.e}{\simeq} 0$.

Contract. $\Rightarrow$ exact since $h^i(C) \underset{0}{\overset{id}{\rightrightarrows}} h^i(C)$ are equal $\Rightarrow h^i(C)=0$.

Ex: $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ exact but not contractible. No nonzero maps

$\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$, need for homotopy.

Ex: $C^\cdot = \left(0 \to M \underset{h=id_M}{\overset{id_M}{\rightleftarrows}} M \to 0 \right)$ contractible $\forall M \in \mathcal{A}$.

Thm: Every contractible complex is isom to $\bigoplus \left(0 \to M \xrightarrow{id} M \to 0\right)$
(w/ various shifts)

i.e. $0 \to M \to M \to 0$ is the only indecomp contractible complex (when M indecomp)

Pf: $C^\cdot$ contractible $\Rightarrow$ $1_C = dh+hd$ for some h. (fix it)

Then $dhd = d(1-dh) = d$. So $(dh)^2 = dh$, it's an idempotent!

$I = dh+hd$ is a sum of two orthogonal idempotents.

Let $A^i := \mathrm{Im}(hd) \subset C^i \supset \mathrm{Im}(dh) =: B^i$. $C^i = A^i \oplus B^i$

1) $d|_B = 0$ since $ddh=0$.

$$C^\cdot = \left(\begin{tikzcd} A^{i-1} \arrow[r, "\rho"] \arrow[dr, "\varphi"] & A^i \arrow[r, "\rho"] \arrow[dr, "\varphi"] & A^{i+1} \\ \oplus & \oplus & \oplus \\ B^{i-1} & B^i & B^{i+1} \end{tikzcd} \right)$$

2) $d^2=0 \Leftrightarrow$ $\rho^2=0$, $\varphi\rho=0$ $\Leftrightarrow d_\rho=0$

But then $p = (dh + hd)p = hdp = 0$

(since $dh = 0$ on A)

Since $dh|_B = id_B$ can deduce φ is isom w/ inverse is on $h|_B$.

So
$$C^\bullet = \left(\begin{tikzcd}[column sep=small] & A^{i-1} \arrow[dr, "\sim"] & & A^{i} \arrow[dr, "\sim"] & & A^{i+1} \arrow[dr, "\sim"] & \\ \arrow[r, "\sim"] & \oplus & B^{i} & \oplus & B^{i+1} & \oplus & \\ & B^{i-1} & & B^{i} & & B^{i+1} & \end{tikzcd} \right)$$

$$\cong \bigoplus \left(0 \to A^{i-1} \xrightarrow{\sim} B^{i} \to 0\right)$$

I.e. contractible $\Downarrow$ cone of an isom! Reverse true by G.E.

Called contractible b/c you can contract them away!

Thm (Gaussian Elimination): Suppose inside a complex $X^\bullet$ we have

$$X^\bullet = \left(\cdots \to A \xrightarrow{\begin{bmatrix} d \\ e \end{bmatrix}} \begin{matrix} B \\ \oplus \\ C \end{matrix} \xrightarrow{\begin{bmatrix} \varphi & b \\ a & c \end{bmatrix}} \begin{matrix} B \\ \oplus \\ D \end{matrix} \xrightarrow{\begin{bmatrix} f & g \end{bmatrix}} E \to \cdots \right)$$
with φ an isomorphism

Let $Y^\bullet$ be the same except in these degrees, where

(draw zig zag in other color)

$$Y^\bullet = \left(\cdots \to A \xrightarrow{e} C \xrightarrow{c - a\varphi^{-1}b} D \xrightarrow{g} E \to \cdots \right)$$

Then $X^\bullet \underset{he}{\simeq} Y^\bullet$

Rmk: $0 \to B \xrightarrow{\varphi} B \to 0$ is NOT a subcomplex, or a summand!

Ex: $X^\bullet = \left(\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} \\ \mathbb{C} \arrow[r, "1"'] \arrow[ur, "1"] & \mathbb{C} \end{tikzcd}\right)$ $\quad Y^\bullet = \left(\begin{tikzcd} \mathbb{C} \arrow[dr, "-1"] & \\ & \mathbb{C} \end{tikzcd}\right)$

even though in $X^\bullet$ no map b/w these summands.

Pf Sketch: ① Show $X^\bullet$ is isomorphic to

$$X' = \left(A \xrightarrow{\begin{bmatrix} 0 \\ e \end{bmatrix}} \begin{matrix} B \\ \oplus \\ C \end{matrix} \xrightarrow{\begin{bmatrix} \varphi & 0 \\ 0 & c - a\varphi^{-1}b \end{bmatrix}} \begin{matrix} B \\ \oplus \\ D \end{matrix} \xrightarrow{\begin{bmatrix} 0 & g \end{bmatrix}} E \right) \cong Y \oplus \left(0 \to B \xrightarrow{\varphi} B \to 0\right)$$

this c.o.b. is row reduction, ie Gaussian elimination.

② Use he to contract out $(0 \to B \xrightarrow{\varphi} B \to 0)$, now a summand!

This is such a useful tool in practice!!

In certain circumstances (e.g. R is a local ring) you can keep eliminating until you can't (no part of any differential is an isom $\Leftrightarrow d \in \mathrm{Jac}(R)$)

Result is called a <u>minimal complex</u> and they're unique up to <u>isom</u> of complexes.

Rmk: Knot homology.