Homological Dimension Notes

Def: $\mathcal{A}$ abelian cat. $A \in \mathcal{A}$ nonzero.

If $\mathcal{A}$ has enough proj, define projective dimension $pd(A)$ as $\min_{n \in \mathbb{N} \cup \{\infty\}} \exists\ 0 \to P^{-n} \to \dots \to P^{-1} \to P^0 \to M \to 0$ proj res'n.

" inj, " injective dimension $id(A)$ " $\exists\ 0 \to M \to I^0 \to \dots \to I^n \to 0$

More generally, if P is some class of objects (e.g. free, flat, purple) w/ enough P then the P-dimension $Pd(A)$ = min length of P resolution.

Rmk: $pd(A) = 0 \iff A$ projective. **Rmk:** $pd(0) = -1$ useful convention.

Ex: ① $\mathcal{A} = \mathbb{Z}$-mod

	$\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$	$\mathbb{Q}$
pd	0	1	1
id	1	1	0

$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

② $\mathcal{A} = \mathbb{C}[x]/x^2$-mod

Frobenius

	$\mathbb{C}$	$\mathbb{C}[x]/x^2$
pd = id	∞	0

③ $\mathcal{A} = \mathbb{C}[x]$-mod $\quad pd(\mathbb{C}) = 1 \quad (R \xrightarrow{x} R) \quad id(\mathbb{C})$ not defined, not enough injectives

(Typically when studying rings one just works with R-Mod to ensure enough proj + inj)

④ All quivers in HW examples had $pd(\text{indecomp.}) = \begin{cases} 0 & \text{if projective} \\ 1 & \text{else} \end{cases}$

Lemma: If $Ext^1(P, A) = 0\ \forall A$, then P is projective.

Pf: $B \twoheadrightarrow C \to 0$ WTS $Hom(P, B) \twoheadrightarrow Hom(P, C) \to 0$

Letting A be the kernel, have $Hom(P,B) \to Hom(P,C) \to Ext^1(P,A) \to \dots$ exact. ∎

Lemma: $\mathcal{A}$ has enough proj. Fix $d \in \mathbb{N}$, $A \in \mathcal{A}$. TFAE

(1) For any exact $0 \to M \to \underbrace{P^{-(d-1)} \to P \to \cdots \to P^{-1} \to P^{0}}_{\text{projective}} \to A \to 0$, M is also projective.

(2) $pd(A) \leq d$ (3) $Ext^n(A,B) = 0$ $\forall n > d$, $\forall B$

(4) $Ext^{d+1}(A,B) = 0$ $\forall B$

Pf: Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). When $d=0$, all (1) (2) (3) are asking if A is projective, Lemma says (4) $\Rightarrow$ projective. Now general case, (4) $\Rightarrow$ (1): given sequence, dimensional reduction says $Ext^i(A,B) = Ext^{i-d}(M,B)$ $\forall B$. So (4) $\Rightarrow$ $Ext^1(M,B) = 0$ $\forall B$ $\Rightarrow$ M is projective $\Rightarrow$ (1). $\boxtimes$

Rmk: Similar arguments show: if $0 \to K \to \underset{proj}{P} \to A \to 0$ then $pd(K) = pd(A) - 1$ or $pd(A) = 0$ $pd(A)$

$\Uparrow$

Exercise: $0 \to A \to B \to C \to 0$ then $pd(B) = \max\{pd(A), pd(C)\}$ unless $pd(C) = pd(A) + 1$, in which case $pd(B) \leq \max\{pd(A), pd(C)\} = pd(C)$, but could be much less. Ex: $0 \to K \to P \to C \to 0$ $pd(P) = 0$, much less

Lemma: Same as lemma above, but swap proj $\Leftrightarrow$ inj, $Ext(A,B) \leftrightarrow Ext(B,A)$, etc.

Thm: Spose $\mathcal{A}$ has enough proj + inj. Then

$$\sup_{A \in \mathcal{A}} \{pd(A)\} = \sup_{A \in \mathcal{A}} \{id(A)\} = \sup_{d \in \mathbb{N}} \{Ext^d(A,B) \neq 0 \text{ for some } A, B\}$$

$\boxed{\text{(possibly } \infty)}$

This is called $gldim(\mathcal{A})$ global dimension.

Pf: Immediate from (2) $\Leftrightarrow$ (3) above.

Notation: R a ring. $lgldim(R) = gldim(R\text{-}Mod)$, $rgldim(R) = gldim(Mod\text{-}R)$.

Some examples we'll see:

comm. ring

we won't see this example; go take Rep theory.

ring	$\mathbb{Z}$	$\Bbbk$ field	$\Bbbk[x_1,\dots,x_n]$	$R[x_1,\dots,x_n]$	reps of quiver	$\mathcal{O}_{\mathfrak{sl}_n}$
gldim	1	0	n	$\mathrm{gldim}(R)+n$	1	$n(n-1)$

(Thinks: If $\mathrm{gldim}(A)=n$, wouldn't it be great to have a machine for producing length n projective resolutions?? Canonically??? Soon, for special cases.)

Rings of small hom. dim.

Thm: R a ring then $\mathrm{gldim}(R)=0$ ① $\Leftrightarrow$ R is semisimple ③ $\Leftrightarrow$ everything projective ② ($\Leftrightarrow$ everything injective $\Leftrightarrow$ every ses is split).

Certainly ① $\Leftrightarrow$ ②. ③ depends on how you defined semisimple.

Book's defn: R is semisimple if all left ideals of R are summands, i.e. $0\to I\to R\to R/I\to 0$ splits.

Certainly I injective or R/I projective $\Rightarrow$ it splits. It splits $\Rightarrow$ I, R/I projective b/c summand of free.

Baer's Criterion: $E\in R\text{-Mod}$ is injective $\Leftrightarrow$ $\forall\, I\subset R$ ideal, the inclusion $I\hookrightarrow R$ induces $\mathrm{Hom}(I,E)\twoheadleftarrow \mathrm{Hom}(R,E)$ (i.e. map from ideals extend to R).

(Rmk: Injectivity is a more general lifting criterion for all inclusions $A\hookrightarrow B$.)

Pf: $\Rightarrow$ clear by remark. $\Leftarrow$: Suppose $0\to A\to B$, $A\to E$. By Zorn's lemma $\exists$ max'l A', $A\subset A'\subset B$ s.t.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & A' \arrow[dl, "\text{extends}"] \\ & E & \end{tikzcd}$$

. If $A'=B$ we win. If $b\in B\setminus A'$, let $I=\{r\in R \mid rb\in A'\}$, an ideal. Get a map $I\to A'$ by acting on b, so $I\to A'\to E$ extends to $R\xrightarrow{\varphi}E$.

Let $A''=\langle A',b\rangle$ and define $A''\to E$ by $b\mapsto\varphi(1)$. It works. ※ ∎

Cor: E injective $\Leftrightarrow$ $\mathrm{Ext}^1(R/I,E)=0$ $\forall\, I\subset R$ ideal.

Cor: $\mathrm{gldim}(R)=\sup\{\mathrm{pd}(R/I)\}$ (not hard) **Cor:** Semisimple $\Rightarrow$ gldim $=0$.

Now for Frob algs:

Prop: Let R be a ring where proj = inj. Then $\mathrm{gldim}(R)=0$ or $\mathrm{gldim}(R)=\infty$.
(and same true for pd of each object)

Pf: If $pd(A)=d\neq 0,\infty$ then $0\to P^{-d}\to P^{-(d-1)}\to\cdots\to P^0\to A\to 0$.
But P^{-d} injective, so first map splits, leaving $0\to K\to P^{-(d-2)}\to\cdots\to P^0\to A\to 0$
but $K\oplus P^{-(d-1)}$ so K proj, so $pd(A)\le d-1$ ※.

Ex: $\Bbbk[G]$ is Frob for any field, finite gp. So
char $\Bbbk\nmid |G| \iff$ semisimple
char $\Bbbk \mid |G| \iff \mathrm{gldim}=\infty$.

Ex: $G=\mathbb{Z}/2\mathbb{Z}$ $\quad \Bbbk[G]=\Bbbk[x]/x^2-1 \xrightarrow{\text{char}\neq 2} \cong \Bbbk[x]/(x+1)(x-1) \underset{\text{CRT}}{=} \Bbbk[x]/(x-1)\times\Bbbk[x]/(x+1)\cong \Bbbk\times\Bbbk$ s.s.

char $=2$: $y=x-1$, $y^2=x^2-1$, so $\Bbbk[G]\cong\Bbbk[y]/y^2=0$ $\quad \mathrm{gldim}=\infty$.

Now dim=1

Def: R is (left) hereditary if every (left) ideal is projective.

Ex: R a PID (like $\mathbb{Z}$, $\Bbbk[x]$) then hereditary since $\underset{(x)}{I}\cong R\cdot x \cong R$ free

Note: $0\to I\hookrightarrow R$ does NOT split.

Thm: R is left hereditary (1) $\iff$ $l\mathrm{gldim}(R)\le 1$ (2) $\iff$ every submod of free is projective (3)

Pf: (2) $\Rightarrow$ (1) b/c $pd(R/I)\le 1$ so I projective by earlier lemma

(2) $\Rightarrow$ (3) b/c $0\to S\to \underset{\text{free}}{F}\to C\to 0$ $\quad pd(C)\le 1 \Rightarrow S$ projective

(3) $\Rightarrow$ (1) clear.

(1) $\Rightarrow$ (2) $\quad pd(R/I)=pd(I)+1$ or $pd(I)=pd(R/I)=0$.
since $pd(I)=0$, $pd(R/I)\le 1$. $\underset{\text{Earlier Cor}}{\Longrightarrow}$ $\mathrm{gldim}(R)\le 1$. ∎

Our big other class of examples will be quiver path algs.

Canonical Resolutions (Not in book in this way)

$S\text{-Mod} = (R,R)\text{-bimod}.$

Spose R is a $\mathbb{k}$-alg for field $\mathbb{k}$. Let $S = R \otimes_{\mathbb{k}} R^{op}$.

Spose we can find one resolution of $R \circlearrowleft S$ by free S-modules.

(These aren't easy to find!!!)

← this is a free R-module, but not a free S-module!

$$\left(\cdots \to S^{\oplus m_1} \to S^{\oplus m_0} \to R \right)$$

Now apply $(\cdot) \otimes_R M$ for any R-module M.

$$\cdots \to (R \otimes_{\mathbb{k}} M)^{\oplus m_1} \to (R \otimes_{\mathbb{k}} M)^{\oplus m_0} \to M$$

Observe: ① still exact! This is because S is free as a right R-module, so $\mathrm{Tor}^1_{R^{op}}(S,M) = 0$, hence $-\otimes M$ will not create cohomology on complexes built from S.

② $R \otimes_{\mathbb{k}} M$ is a free left R-module (just $\dim M$ copies of R)

Thus we get a free resolution of M!! This is functorial in M since $-\otimes M$ is.

Call this a <u>canonical resolution of M</u>, given by an canonical bimodule resolution of R.

<u>Ex 1</u>: $R = \mathbb{C}[x]$ $S = \mathbb{C}[y,z]$ where $y = x \otimes 1$ $z = 1 \otimes x$ $(y,z \mapsto x)$

As an S-module, $R \cong S/(y-z)$ so $\left(S \xrightarrow{y-z} S \right) \to R$ is can. resn.

Thus for any R-module,

think of as $\mathbb{C}[y]$

$$0 \to \mathbb{C}[x] \otimes_{\mathbb{C}} M \xrightarrow{\varphi} \mathbb{C}[x] \otimes_{\mathbb{C}} M \xrightarrow{\psi} M \to 0 \quad \text{is can res}$$

ψ is just mult, b/c $y \mapsto$ action of x. $f \otimes m \mapsto fm$.

φ is mult by $y-z$, i.e. $f \otimes m \mapsto xf \otimes m - f \otimes xm$.

(Hence $\mathrm{gldim}(\mathbb{C}[x]) \leq 1$.) (<u>Note</u>: $\mathbb{C}[x] \otimes M$ is f.g. $\Leftrightarrow$ M is f.d.)

Ex 2: The useless yet universal example: the bar resolution. Every $\mathbb{k}$-alg has a free bimodule resolution as follows: (work right to left)

$$\cdots \to R\otimes R\otimes R\otimes R \to R\otimes_{\mathbb{k}} R\otimes_{\mathbb{k}} R \to R\otimes_{\mathbb{k}} R \xrightarrow{\text{mult}} R \to 0$$

$$f\otimes g\otimes h \mapsto fg\otimes h - f\otimes gh$$

$$d(a_0\otimes\cdots\otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_n$$

(Think: simplicial homology)

Of course, this is far from being minimal or good for computing gldim, but you can use it for certain formal arguments.

Ex 3: Q a quiver. Recall $A=\mathbb{k}[Q]$ the path alg. $e_j\mathbb{k}[Q]e_i$ = paths from i to j.

We'll work w/ projective res'ns rather than free, you'll see why its ok.

Here's the canonical resolution.

$$0 \to \bigoplus_{\substack{x\in E \\ \text{edges} \\ i\xrightarrow{x} j}} Ae_j\otimes e_iA \to \bigoplus_{\substack{i\in V \\ \text{vertices}}} Ae_i\otimes_{\mathbb{k}} e_iA \xrightarrow{\text{compose}} A \to 0$$

$$fe_j\otimes e_ig \mapsto fxe_i\otimes e_ig - fe_j\otimes e_jxg$$

These summands are projective $A\otimes A^{op}$-modules (for the idempotents $e_j\otimes e_i$) and projective as right or left A-modules (many copies of Ae_j or of e_iA)

so ① $-\otimes_A M$ still exact b/c $\mathrm{Tor}^{>0}(e_iA, M)=0$

② $Ae_j\otimes_{\mathbb{k}} e_iM$ is still proj as left A-module. (just $\dim e_iM$ copies of Ae_j) for any A-module M.

Thus our argument still works to get a can. res'n of M for any A-module M.

Thm $\mathrm{gldim}(A)=1$ (unless no edges, then $\mathrm{gldim}(A)=0$ semisimple)

Explicitly, if $M_i=e_iM$ then $\bigoplus_x Ae_j\otimes M_i \to \bigoplus_i Ae_i\otimes M_i \to M \to 0$

Ex: $\circ \longrightarrow \circ \xrightarrow{x} \circ$ $\quad M = 0 \to \mathbb{C} \to 0$ $\quad M_2 = \mathbb{C}$ $\quad M_1 = M_3 = 0$

then $\bigoplus Ae_i \otimes M_i = Ae_2 \otimes M_2 = Ae_2 = \underset{e_2Ae_2}{0 \to \mathbb{C}} \xrightarrow{\sim} \underset{e_3Ae_2}{\mathbb{C}}$

$\bigoplus\limits_{x} Ae_j \otimes M_i = Ae_3 \otimes M_2 = Ae_3 \quad 0 \to 0 \to \mathbb{C}$

the map $Ae_3 \to Ae_2$ is composition w/ x, i.e.

$$\begin{array}{ccccc} 0 & \to & 0 & \to & \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow s \\ 0 & \to & \mathbb{C} & \xrightarrow{\sim} & \mathbb{C} \end{array} .$$

Another major source of canonical resolutions: Koszul complexes.

This means two different things in two different contexts! Don't confuse!!!

① Ksz cx of regular sequence
↑ I'll do this now

② Ksz cx of positively graded algebra
↑ additional topic.

Let S be a commutative ring. Interested in resolving $S/(x_1, x_2, \ldots, x_n)$ for $\vec{x} = (x_1, \ldots, x_n) \in S$
↑ ordered sequence

(Example: $R = \mathbb{C}[x_1 \ldots x_n]/I$ $\quad S = R \otimes R^{op} = \mathbb{C}[y_1 \ldots y_n, z_1 \ldots z_n]/I_y, I_z$
then as R-bimodule, i.e. S-module, $R \cong S/(y_1 - z_1, y_2 - z_2, \ldots, y_n - z_n)$)

If $r = 1$, $0 \to S \xrightarrow{x_0} S \to S/x_0 \to 0$ is exact $\iff$ x_0 is a nonzero divisor nzd.

$n = 2$
$\vec{x} = (x, y)$ $\quad 0 \to S \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} \begin{matrix} S \\ \oplus \\ S \end{matrix} \xrightarrow{[x\ y]} S \to S/(x,y) \to 0$ is exact $\Longleftarrow$ x is nzd, $\bar{y}$ is nzd in $S/(x)$.
or vice versa.

why? $\mathrm{Ker}\left(S \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} \begin{matrix} S \\ \oplus \\ S \end{matrix}\right) = \left\{ f \;\middle|\; \begin{matrix} xf = 0 \\ -yf = 0 \end{matrix} \right\} = 0$

$\mathrm{Ker}\left(\begin{matrix} S \\ \oplus \\ S \end{matrix} \xrightarrow{[x,y]} S\right) = \{(g,h) \mid xg + yh = 0\}$ in $S/(x)$ have $\bar{y}\bar{h} = 0$
$\Rightarrow \bar{h} = 0 \Rightarrow h = xk.$

so $x(g + yk) = 0 \Rightarrow g + yk = 0 \Rightarrow g = -ky.$

Thus (g,h) is image of k via $\begin{bmatrix} -y \\ x \end{bmatrix}$.

<u>Def</u>: The <u>Koszul complex</u> $Kz(S;\vec{x})$ is $\bigotimes_{i=1}^{r}\left(S \xrightarrow{x_i} \underline{S}\right)$.

We haven't done $\otimes$ of complexes yet, so here's explicit version.

In hom degree $-i$ have $S^{\oplus\binom{r}{i}}$ indexed by $I\subset\{1,\dots,r\}$ of size i.

Call this summand S_I, i.e. $\bigoplus_{\substack{I\\ |I|=i}} S_I$ is in hom deg $-i$.

The differential sends S_I to $\bigoplus_{j\in I} S_{I\setminus j}$ via mult by $(-1)^k x_j$ where $k=\#(\{1,\dots,j-1\}\cap I)$

Signs make $d^2=0$.

<u>Ex</u>:

$$\begin{tikzcd}
 & & S_{12} \arrow[r, "-x_2"] \arrow[dr, "x_1"] & S_1 \arrow[dr, "x_1"] & & & \\
0 \arrow[r] & S_{123} \arrow[ur, "x_3"] \arrow[r, "-x_2"] \arrow[dr, "x_1"] & S_{13} \arrow[ur, "-x_3"] \arrow[dr, "x_1"] & S_2 \arrow[r, "x_2"] & \underline{S_{\emptyset}} \arrow[r] & S/(x_1,x_2,x_3) \arrow[r] & 0 \\
 & & S_{23} \arrow[ur, "-x_3"] \arrow[r, "x_2"] & S_3 \arrow[ur, "x_3"] & & &
\end{tikzcd}$$

Clearly $h^0(Kz)=S/(x_i)$, but rest of cohom not so clear, not usually exact!

<u>Def</u>: An ordered sequence $\vec{x}$ is <u>regular</u> if x_i is a nzd in $S/(x_1,\dots,x_{i-1})$

<u>Thm</u>: $h^i(Kz(S;\vec{x}))=0$ $\forall i<0$ if $\vec{x}$ is regular. <u>Pf</u>: Later.

<u>Rmks</u>: ① Not iff. In fact, $Kz(S,\vec{x})\cong Kz(S,\vec{x}')$ if $\vec{x}'$ a permutation of $\vec{x}$, but regularity is NOT preserved by permutation!

<u>Ex</u>: $S=\mathbb{C}[x,y,z]$ $\quad \vec{x}=(x, y(1-x), z(1-x))$ regular

$\vec{x}'=(z(1-x), y(1-x), x)$ not regular

② R local + Noeth, or R graded in degrees ≥ 0 and x_i all homogeneous w/ $\deg x_i>0$

$\Longrightarrow$ permutation of regular is regular.

③ $S=\mathbb{C}[y_1,\dots,y_r,z_1,\dots,z_r]$ $\quad \vec{x}=(y_1-z_1,\dots,y_r-z_r)$ is regular, so Kz gives canonical bimodule resolution of $R=\mathbb{C}[x_1,\dots,x_r]$. gldim $=r$.

(4) $S=\mathbb{C}[x_1,\dots,x_r]$ $\quad \vec{x}=(x_1,\dots,x_r)$ regular $\quad S/(\vec{x})=\mathbb{C}$.

so get
$$\left(S \xrightarrow{x_1,\,x_2,\,\dots} S\oplus\cdots\oplus S \to \cdots \to S\right) \twoheadrightarrow \mathbb{C}$$
exact.

To compute $\mathrm{Ext}^*(\mathbb{C},\mathbb{C})$, apply $\mathrm{Hom}(-,\mathbb{C})$ to get
$$\left(\mathbb{C} \xleftarrow{0} \mathbb{C}\oplus\cdots\oplus\mathbb{C} \quad\cdots\quad \mathbb{C}\oplus\cdots \xleftarrow{0} \mathbb{C}\right)$$

all differentials 0. i.e.
$\dim \mathrm{Ext}^i(\mathbb{C},\mathbb{C}) = \binom{r}{i}$
i.e. $\mathrm{Ext}^*(\mathbb{C},\mathbb{C}) \cong \Lambda^*(\mathbb{C}^r)$ as v.s.

Soon we'll see that $\mathrm{Ext}^*_S(\mathbb{C},\mathbb{C})$ is a graded algebra, and this is iso of algebras!!!

Baby example of Koszul duality b/w positively graded algebras.

Note: $\mathrm{Ext}^*_\Lambda(\mathbb{C},\mathbb{C}) \cong S$ as graded algebras!!