Koszul Redux To make the next proof cleaner, I'm going to use the opposite sign convention for Ksz!! It's worth it.

Recall $\mathrm{Ksz}(R,\vec{x}) =$ $\boxed{n=3}$

$$\left(\begin{tikzcd} & R_{23} \arrow[r, "-x_2"] \arrow[dr, "x_3"] & R_3 \arrow[dr, "x_3"] & \\ R_{123} \arrow[ur, "x_1"] \arrow[r, "-x_2"] \arrow[dr, "x_3"] & R_{13} \arrow[ur, "-x_1"] \arrow[dr, "x_3"] & R_2 \arrow[r, "x_2"] & R_\emptyset \\ & R_{12} \arrow[ur, "-x_1"] \arrow[r, "x_2"] & R_1 \arrow[ur, "x_1"] & \end{tikzcd}\right)$$

sign: $R_I \xrightarrow{(-1)^k x_j} R_{I \setminus j}$ $\quad k = \#\{i \in I \mid i > j\}$

Thm: $\vec{x}$ is regular $\Rightarrow$ $\mathrm{Ksz}(R,\vec{x})$ is exact, except at $h^0(\mathrm{Ksz}) = R/(\vec{x})$.

Pf: Induct. $n=1$: $\quad R \xrightarrow{x_1} R$ injective $\Leftrightarrow$ x_1 is a nzd. $\boxed{\checkmark}$

- Let $K^\circ = \mathrm{Ksz}(R,(x_1 \ldots x_{n-1}))$. In fact, K appears twice in $\mathrm{Ksz}(R,\vec{x})$, see red above. One is K, other is $K[1]$ (note negated differentials!)

We see $\quad \mathrm{Ksz} = \mathrm{Cone}(K \xrightarrow{x_n} K)$

Recall: $\mathrm{Cone}(A \xrightarrow{f} B)$ is
$$\begin{matrix} B \\ \oplus \\ A[1] \end{matrix} \xrightarrow{\begin{pmatrix} d_B & f \\ 0 & d_A \end{pmatrix}} \begin{matrix} B \\ \oplus \\ A[1] \end{matrix}$$

i.e. $\exists$ s.e.s. $\quad 0 \to K \to \mathrm{Ksz} \to K[1] \to 0$

$\Rightarrow \exists$ l.e.s.

$$\underset{\substack{\cong \\ h^i(K)}}{h^{i-1}(K[1])} \xrightarrow{\delta_i} h^i(K) \to h^i(\mathrm{Ksz}) \to \underset{\substack{\cong \\ h^{i+1}(K)}}{h^i(K[1])} \xrightarrow{\delta_{i+1}} h^{i+1}(K) \to$$

Claim: δ is induced by multiplication by $x_n : K \to K$! $\quad$ This is a fact for any cone. More next week.

So get ses

$$0 \to \operatorname{Coker}\delta_i \to h^i(Ksz) \to \operatorname{Ker}\delta_{i+1} \to 0$$

This helps you compute even when $\vec{x}$ is not regular.

When $\vec{x}$ regular: $h^i(K)=0$ for $i>0$, so $h^i(Ksz)=0$ for $i>1$.

$$h^{-1}(Ksz) \cong \operatorname{Ker}\left(h^0(K) \xrightarrow{x_r} h^0(K)\right) = \operatorname{Ker}\left(R/(x_1 \ldots x_{r-1}) \xrightarrow{x_r} R/(x_1 \ldots x_{r-1})\right)$$

$= 0$ since x_r nzd mod the previous.

$$h^0(Ksz) = \operatorname{Coker}\left(\longrightarrow\right) = R/(\vec{x}).$$ ∎

Koszul cx of a module $\vec{x} \in R$ commutative, $M \in R$-Mod then

$$Ksz(M,\vec{x}) = \left(M_{123} \rightrightarrows M_{23} \cdots \rightrightarrows \underline{M_\emptyset}\right)$$

> **Warning:** It is NOT the canonical resolution of M...
>
> That was like $R \otimes_k R \otimes_k M \to R \otimes_k M$

Clearly $h^0 = M/(x_i M)$, but what are higher cohom?

This is $Ksz(R,\vec{x}) \otimes_R M$, so it should compute $\operatorname{Tor}^i_R(R/(\vec{x}), M) \cong h^{-i}(Ksz(M,\vec{x}))$ when $\vec{x}$ regular.

↑ a free res'n of $R/(\vec{x})$ when $\vec{x}$ is regular

But what about $\operatorname{Hom}(R/(\vec{x}), M) = \{m \in M \mid x_i m = 0 \ \forall i\}$

Compute the derived functor via $\operatorname{Hom}(Ksz(R,\vec{x}), M)$

$$\| $$

$$\left(M_{123} \leftleftarrows \cdots \leftleftarrows \underline{M_\emptyset}\right) \cong Ksz(M,\vec{x})^{[-r]}.$$

So **Prop:** $\operatorname{Ext}^i(R/(\vec{x}), M) \cong \operatorname{Tor}^{r-i}(R/(\vec{x}), M)$ when $\vec{x}$ regular.

Koszul cohomology vs Koszul homology

<u>Special case</u>: R a comm. ring. $\exists$ functors $(R,R)\text{-bim} \overset{HH_0}{\underset{HH^0}{\rightrightarrows}} R\text{-mod}$

$HH_0(M) = M/(xm - mx)$ quotient where left-right actions agree

$HH^0(M) = \{m \in M \mid xm = mx\}$ sub ——//——

Let $S = R \otimes R^{op}$, I the ideal generated by $(x \otimes 1 - 1 \otimes x)_{x \in R}$ so that

$R \cong S/I$ as S-mod. Then $HH_0(M) = R \otimes_S M$ has higher derived $HH_i = \mathrm{Tor}_i$

$HH^0(M) = \mathrm{Hom}_S(R, M)$ $\quad HH^i = \mathrm{Ext}^i$

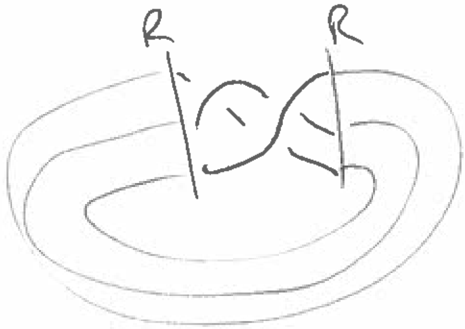
When I is cut out by a regular sequence (e.g. as we saw for $R = \mathbb{C}[x_1, \ldots, x_r]$)

then $HH^i(M) \cong HH_{r-i}(M)$

Hochschild cohom vs. hom.

Useful property: $HH_i(M \otimes N) \cong HH_i(N \otimes M)$ as v.s. (NOT as R-modules)

(outside action vs inside action) so often appears when we want to glue a strip into a cylinder.



If have an R-bimodule for a braid, get a v.s. for a braid closure.

<u>Eisenbud's Thm on periodic resolutions</u> $R = \mathbb{C}[x_1, \ldots, x_n]$ $\quad H = \mathbb{C}[x_1, \ldots, x_n]/(f)$ "hypersurface ring"

<u>Thm</u>: $M \in H\text{-mod}$. Then M has a free H-resolution which is eventually 2-periodic.

<u>Pf; Sketch</u>: M is an R-module too. Using the Canonical resolution, get

$0 \to F^{-n} \to \cdots \to F^0 \to M \to 0$ free R-mod resolution, Now $\otimes H$ to get

$$0 \to P^{-n} \to \cdots \to P^0 \to M \to 0 \qquad P^i \text{ are free } H\text{-modules.}$$

but is it exact? No, it computes $\mathrm{Tor}_R^i(R/(f), M)$ using a proj res of M.

Can also compute this w/ proj resolution of $R/(f)$! $\quad 0 \to R \xrightarrow{f} R \to R/(f) \to 0$

so $\mathrm{Tor}^i = 0$ for $i > 1$. $\qquad \otimes M: \quad 0 \to M \xrightarrow{f} M \to M/(f) \to 0$

$$\mathrm{Tor}^i(R/(f), M) = \begin{cases} M & i = 0, 1 \\ 0 & \text{else} \end{cases}$$

but $f = 0$ on M, so $\quad 0 \to M \xrightarrow{0} M \xrightarrow{\sim} M \to 0$

Now $M \cong \mathrm{Ker}(P^{-1} \to P^0) / \mathrm{Im}(P^{-2} \to P^{-1})$ so

$$\begin{tikzcd} & P^0 \arrow[d] \arrow[dl, dashed] \\ \mathrm{Ker}(P^{-1} \to P^0) \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

. Continue using projective lifting to construct

$$\begin{tikzcd} 0 \arrow[r] & P^{-n} \arrow[r] \arrow[d] & \cdots & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[d] \\ 0 \arrow[r] & P^{-n} \arrow[r] & \cdots \, P^{-2} \arrow[r] & P^{-1} \arrow[r] & P^0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

looking at h^{-1}, get zero (the M on top cancels the M on bottom)

but now $h^{-2} = M$ again! Repeat!

$$\cdots \to \begin{matrix} P^0 \\ \oplus \\ P^{-2} \\ \oplus \\ P^{-4} \end{matrix} \to \begin{matrix} P^{-1} \\ \oplus \\ P^{-3} \end{matrix} \to \begin{matrix} P^0 \\ \oplus \\ P^{-2} \end{matrix} \to P^{-1} \to P^0 \to M$$

eventually 2-periodic because n is finite so eventually nothing new is added.

If time: First change of rings thm $\quad f \in R$ a nzd, central, M an $R/(f)$ mod

w/ $\mathrm{pd}_{R/f}(M) < \infty$ then $\mathrm{pd}_R(M) = \mathrm{pd}_{R/f}(M) + 1$. $\quad$ Use a similar argument.