Projectives Etc

Today $\mathcal{A} \subset R\text{-Mod}$ closed under kernels/cokernels. Later: $\mathcal{A}$ abelian.

Defi: P is projective if

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \\ M \arrow[r, two heads] & N \arrow[r] & 0 \end{tikzcd}$$

I is injective if

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, hook] \arrow[d] & N \arrow[dl, dashed, "\exists"] \\ & I \end{tikzcd}$$

Prop: P projective $\Rightarrow$ $0 \to A \to B \to P \to 0$ is split

I inj $\Rightarrow$ $0 \to I \to B \to A \to 0$ is split.

Pf:

$$\begin{tikzcd} & P \arrow[dl, dashed, "q"'] \arrow[d, "id"] \\ B \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

$gq = id_P \Rightarrow$ split.

Cor: In R-Mod (resp. R-mod), P is Proj $\Leftrightarrow$ $P \overset{\oplus}{\subset} \underbrace{R^{\oplus X}}_{\text{free module}}$ ← some index set (finite)

Ex: $\mathbb{C}[x]\text{-mod}_{fd}$ has no projectives (except zero)!

Everything has extensions — make Jordan block bigger!

Ex: $\mathbb{C}[x]/x^3$ has free module $M_{\begin{pmatrix} 0&1&\\&0&1\\&&0\end{pmatrix}}$ projective.

But $M_{\begin{pmatrix} 0&1&\\&0&1\\&&0\end{pmatrix}}$ is not proj as $\mathbb{C}[x]/x^4$-module since $0 \to M_{(0)} \to M_{\begin{pmatrix} 0&1&&\\&0&1&\\&&0&1\\&&&0\end{pmatrix}} \to M_{\begin{pmatrix} 0&1&\\&0&1\\&&0\end{pmatrix}} \to$

not a module for $\mathbb{C}[x]/x^3$

Moral: Projectivity (+ extensions) depend on context not on underlying v.s. w/ operators.

Can have $\mathcal{A} \underset{\text{full subcat}}{\subset} \mathcal{B}$, $P \in \text{Proj}\,\mathcal{A}$ but $P \notin \text{Proj}\,\mathcal{B}$.

Prop: R Noeth + local then projective f.g. $\Leftrightarrow$ free f.g.

Ex: $\mathbb{C}[x]/x^3$. Non-ex: $\mathbb{C}[x]/x^2-x = \left(\mathbb{C}[x]/x\right) \times \left(\mathbb{C}[x]/(x-1)\right)$

Prop: Over $\mathbb{Z}$, injective $\Leftrightarrow$ divisible i.e. $\forall_{\substack{m \in M \\ k \in \mathbb{Z}}} \exists_{n \in M}$ s.t. $kn = m$.

$\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, but NO f.g. ab gps are divisible!

$\mathbb{Z}$-mod has no injectives.

Prop: Let R be a Frobenius algebra (later in course). Then proj $\Leftrightarrow$ inj.

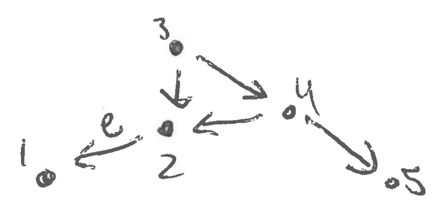
Ex: Cohomology rings of smooth compact manifolds.

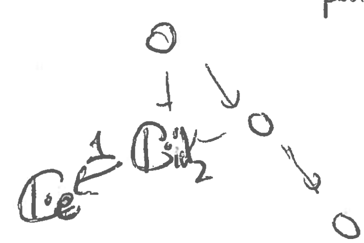
$$\mathbb{C}[x]/x^d = H^*(\mathbb{P}^{d-1}_{\mathbb{C}}) \qquad \Lambda^* \mathbb{C}^n = H^*(\overbrace{S^1 \times \dots \times S^1}^{n})$$

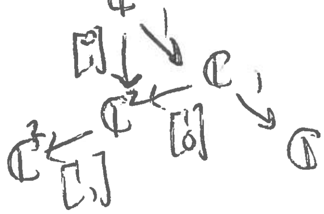
Prop: In Q-rep, $\{\text{indecomp. proj}\}/\cong \;\longleftrightarrow\; V$ ← vertices

$$P_i \longleftrightarrow i$$

$$(P_i)_j = \mathbb{C} \cdot \langle \text{paths } i \to j \rangle$$ w/ edge action being post-comp. of paths

Ex: 

P_2 

P_3 

Ex: $\overset{1}{\bullet} \to \overset{2}{\bullet}$

$P_2 = 0 \to \mathbb{C}$ projective

$P_1 = \mathbb{C} \xrightarrow{1} \mathbb{C}$ proj

whereas $S_1 = \mathbb{C} \to 0$ not projective b/c of nonsplit

$$0 \to P_2 \to P_1 \to S_1 \to 0$$

Interesting exercise: Compute $\mathrm{Hom}(P_i, P_j)$.

Aside from these examples: finding proj/inj is hard.

For ∞-dim algs, injectives can be very hard to find, not f.g. often

For other abelian cats may not exist. ~~Sheaves~~ Ex: Sheaves of v.s. have injectives but not projectives

Def: $\mathcal{A}$ has enough projectives if $\forall M \in \mathcal{A}$ $\exists P \twoheadrightarrow M$ w/ P projective

— // — injectives $\exists M \hookrightarrow I$ w/ I injective

Ex: R-mod has enough projectives but often not enough injectives.

R-Mod has both.

$\mathbb{C}[x]\text{-mod}_{fd}$ has neither.

$\mathbb{C}[x]\text{-mod}_{diag}$ is semisimple, has both. ← x acts diagonalizably.

not "the"

Def: A <u>projective resolution</u> of $M \in \mathcal{A}$ is a complex

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{\varepsilon} M \to 0$$

which is exact, w/ P^i projective $\forall i \geq 0$.

(P°) represents $\to P^{-2} \to P^{-1} \to P^0 \to 0$ which is not exact.

$$h^i(P^\circ) = \begin{cases} 0 & i \neq 0 \\ M & i = 0 \end{cases}$$

but $(P^\circ) \to \underline{M}$ is ~~a~~ qison, i.e.

$$\begin{array}{ccccccccccc} \cdots \to & P^{-2} & \to & P^{-1} & \to & P^0 & \to & 0 & \to & 0 & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow \varepsilon & & & & & \\ \cdots \to & 0 & \to & 0 & \to & M & \to & 0 & \to & 0 & \to \cdots \end{array}$$

i.e. <u>Proj resolution of M</u> is a pair (P°, ε) $\quad P^\circ$ a complex of projectives, ε a qison $(P^\circ) \to \underline{M}$.

An <u>injective resolution</u> of M is $0 \to M \to I^0 \to I^1 \to \cdots$
w/ I^i injective $\forall i \geq 0$.

<u>Lemma:</u> If $\mathcal{A}$ has enough projectives then every M admits a proj resolution.

<u>Pf:</u> Choose $P^0 \overset{\varepsilon}{\twoheadrightarrow} M$. Choose $P^{-1} \overset{d_{-1}}{\twoheadrightarrow} \operatorname{Ker} \varepsilon$.
Choose $P^{-2} \overset{d_{-2}}{\twoheadrightarrow} \operatorname{Ker} d^{-1}$. Etc.

Finding proj resolns is the bread & butter of Hom Alg.

Ex 1: $R = \mathbb{C}[x]/x^3$ free rank 1 R-module is $M\left(\begin{smallmatrix}\circ & & \\ & \circ & \\ & & \circ\end{smallmatrix}\right)$, also inj b/c Frobenius

$$\cdots \to R \xrightarrow{x^2} R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} R \to M_{(\circ)} \to 0$$
$$x \mapsto 0$$

Rmk: There is no finite proj resolution. Hom dim, later.

Ex 2: $\mathbb{Q}$-rep for $\bullet \to \bullet$

$$0 \to P_2 \to P_1 \to S_1 \to 0$$

also silly things like $0 \to P_1 \xrightarrow{id} P_1 \to 0$

Oops

Inj resolut.

$$0 \to M_{(\circ)} \xrightarrow{x^2} R \xrightarrow{x} R \xrightarrow{x^2} R \xrightarrow{x} \cdots$$

Ex 3: $R = \mathbb{C}[x]$

$$0 \to R \xrightarrow{x} R \to M_{(\circ)} \to 0$$

Ex 3': $R = \mathbb{C}[x,y]$

$M = \mathbb{C}[x,y]/(x,y)$

$$\left(R \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} R \oplus R \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \right) \to M \to 0$$

Koszul complex later.

Lots of exercises.