Lecture 2 Baby Rep Theory

What I care about when playing with additive/abelian categories. Since we haven't defined them yet, we'll stick with R-Mod + related categories.

Def: Let R be a ring. A nonzero R-module M is <u>simple</u> or <u>irreducible</u> if it has no submodules which are proper (nonzero, not everything). (Equiv, no proper quotients.)

M is <u>indecomposable</u> if it has no proper <u>summands</u> $\leftarrow$ a sub w/ a complement.

For $A \subset M$, $\quad A \overset{\oplus}{\subseteq} M$ if $\exists\, B \subset M$ s.t. $\quad A+B=M \quad A \cap B = \{0\}$

Ex 1: $R=\mathbb{C}$. $\quad$ Simple $\overset{(*)}{\Longleftrightarrow}$ indecomp $\Longleftrightarrow$ 1 dim. $\qquad (*)$ true $\Longleftrightarrow R$ is <u>semisimple</u>.

Ex 2: $R=\mathbb{C}[x]$ $\qquad \mathcal{C} = R\text{-Mod}_{fd}$ = cat. of R-modules which are f.d. as V.S.

⚠ Very different from R-Mod!!

Equiv, an object of $\mathcal{C}$ is a pair (V, x) $\quad V$ a f.d. $\mathbb{C}$ v.s., $x \in \mathrm{End}(V)$

A morphism $\varphi: V \to W$ is
$$\begin{tikzcd} V \arrow[r, "x"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\varphi"] \\ W \arrow[r, "x"] & W \end{tikzcd}$$

Note: φ invertible on underlying v.s.
$\Rightarrow \varphi^{-1}$ also commutes w/ x
so φ invertible in $\mathcal{C}$.

My notation: $M_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ is object where $M=\mathbb{C}^2$ + x is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

even lazier: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\qquad$ Of course, $M_A \cong M_{A'}$ when $A \sim A'$!

Submodule: subspace preserved by x. $\qquad \begin{pmatrix} \overbrace{*}^{\text{submodule}} & * \\ 0 & * \end{pmatrix}$

Summand: $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$

Any f.d. $\mathbb{C}[x]$ v.s. ~~w~~ w/ operator X has an eigenvector, spanning 1D submod. ②

$\Rightarrow$ all simples are 1D. $\{\text{Simples}\} \leftrightarrow \mathbb{C}$

$M_{(\lambda)} \leftarrow\!\!\shortmid \lambda$

Consider $M_{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}$. All proper submodules contain $\langle e_1 \rangle$, so no two are transverse!

M is indecomposable!

Jordan Normal Form: Up to isom, $X = \begin{pmatrix} \square & & 0 \\ & \ddots & \\ 0 & & \square \end{pmatrix}$ and blocks are $\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$

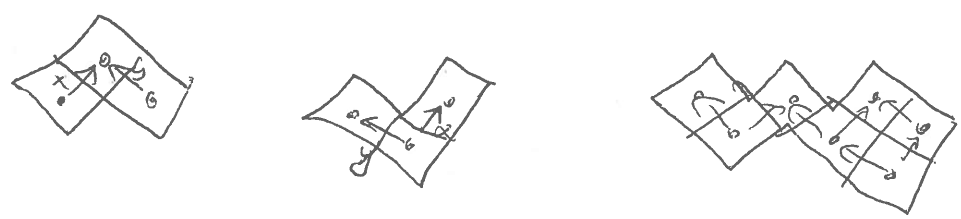
i.e. $M = \bigoplus M_{\begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}}$ $\leftarrow$ these are indecomposable

$\{5 \text{ dim indecomposables}\} \leftrightarrow$ some reasonable moduli space

---

Ex 3: $R = \mathbb{C}[x,y]$ $\mathcal{C} = R\text{-mod}_{\text{f.d.}}$ B/c x,y commute, can find simultaneous eigenvector $\Rightarrow$ all simples 1D

Indecomposables?



...

Ex 4: $R = \mathbb{C}\langle x,y \rangle$ $\mathcal{C} = R\text{-mod}_{\text{f.d.}}$

Simples? Indecomp? Moduli space is garbage!

Don't study all rings. Proving a ring has nice module theory is interesting!

Def: Short exact sequence $0 \to B \xrightarrow{f} M \xrightarrow{g} A \to 0$

i.e. f injective, A "=" coker f, g "=" coker f $\Leftrightarrow$ g surjective, $B = \ker g$, $f = \ker g$

Say M is extension of A by B. $\boxed{\text{Alphabetical: ext., quot., sub.}}$

Ex 2:

$$0 \to M_{(2)} \xrightarrow{\binom{1}{0}} M_{\begin{pmatrix}2&1\\0&2\end{pmatrix}} \to M_{(2)} \to 0$$

$$0 \to M_{(2)} \xrightarrow{\begin{pmatrix}1\\0\\0\end{pmatrix}} M_{\begin{pmatrix}2&1&\\&2&1\\&&2\end{pmatrix}} \to M_{\begin{pmatrix}2&1\\&2\end{pmatrix}} \to 0$$

$$0 \to M_{\begin{pmatrix}2&1\\&2\end{pmatrix}} \xrightarrow{\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}} M_{\begin{pmatrix}2&1&\\&2&1\\&&2\end{pmatrix}} \to M_{(2)} \to 0$$

all non-trivial... or are they?

Ex (4)

$$0 \to B \xrightarrow{\binom{0}{id_B}} A \oplus B \xrightarrow{[id_A\ 0]} A \to 0 \qquad \underline{\text{trivial}} \text{ s.e.s.}$$

Def: A s.e.s. is trivial if it is isomorphic to (4) relative to A and B, i.e.

$$\begin{array}{ccccccccc} 0 & \to & B & \to & M & \to & A & \to & 0 \\ & & \downarrow{\scriptstyle id_B} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle id_A} & & \\ 0 & \to & B & \to & A\oplus B & \to & A & \to & 0 \end{array}$$

We also say the sequence <u>splits</u>.

Thm: TFAE

(1) $0 \to B \xrightarrow{f} M \xrightarrow{g} A \to 0$ is trivial

(2) $\exists\ M \xrightarrow{p} B$ w/ $pf = id_B$

(3) $\exists\ A \xrightarrow{q} M$ w/ $gq = id_A$

(HARDER Miyata '67) If R Noeth, $A, B \in R$-mod then

(4) $M \cong A \oplus B$ ("apparently trivial" split)

Usual proof: (2) ⇒ (1)

$$\begin{array}{ccccccccc} 0 & \to & B & \xrightarrow{f} & M & \xrightarrow{g} & A & \to & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi = \begin{bmatrix} g \\ p \end{bmatrix}} & & \| & & \\ 0 & \to & B & \xrightarrow{i_B} & A\oplus B & \xrightarrow{p_A} & A & \to & 0 \end{array}$$

is a morphism. Is it iso?

$m \in \ker \varphi \Rightarrow m \in \ker g$ ($m \in \ker p$) $\Rightarrow m \in \operatorname{Im} f$, $m = f(n)$

$\Rightarrow n = p(m) = 0 \Rightarrow m = 0$.

Etc to show surj.

THIS PROOF USED ELEMENTS NOT CATEGORY THEORY.

Want: proof using univ. properties of kernels, images, etc. No elements! B/c want same theorem for other abelian categories like sheaves. Need to know univ props... + magical assumptions

Similarly for 5-lemma, Snake lemma, etc. We'll cover this.

Ex 5: Quiver Repns

$\bullet \to \bullet$

$\bullet \circlearrowright$

$\bullet \rightrightarrows \bullet$

$\bullet \to \bullet \leftarrow \bullet \to \bullet$

Def: A quiver $Q = (V, E)$ is a directed graph.

vertices (V), edges (E)

A (f.d.) repn of Q is
- A (f.d.) subspace W_v for each $v \in V$
- A linear map $\varphi_e : W_v \to W_{v'}$ for each $v \xrightarrow{e} v'$ in E.

A morphism is a linear map $\varphi_v : W_v \to W'_v$ for each $v \in V$ making squares commute for each edge.

Subex 1: $Q = \bullet \circlearrowright$ $\quad$ $Q\text{-rep}_{f.d.} = \mathbb{C}[x]\text{-Mod}_{f.d.}$

Def/Thm: Kernels / cokernels / images of morphisms defined termwise.

Subex 2: $Q = \bullet \to \bullet$ $\quad$ Q-rep is $W \xrightarrow{\psi} X$.

Is $W \to 0$ a submod? No, not if ψ is nonzero.

$\operatorname{Ker}\psi \to 0$ is a submod. $\quad$ $0 \to X$ is a submod

If $U \subset W$ then $U \to \psi(U)$ is a submod

Simples: $\mathbb{C} \to 0$

$0 \to \mathbb{C}$

Indecomp: Also have $\mathbb{C} \xrightarrow{1} \mathbb{C}$

anything else?

Prop: $$\text{Indecomp}/\cong \;\overset{\sim}{\longleftrightarrow}\; \left\{ \begin{matrix} 0 \to \mathbb{C} \\ \mathbb{C} \to 0 \\ \mathbb{C} \xrightarrow{\sim} \mathbb{C} \end{matrix} \right\}$$

Pf: $W \xrightarrow{\psi} X$ indecomp. 1) If $\exists\, w \in W$ s.t. $\overset{x}{\overset{\shortparallel}{\psi(w)}} \neq 0$ then

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \mathbb{C} \arrow[r] \arrow[d, "w"'] & \mathbb{C} \arrow[d, "x"] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ W/w \arrow[r] \arrow[d] & X/x \arrow[d] \\ 0 & 0 \end{tikzcd}$$

is it split? Choose $X = \mathbb{C}x \oplus X'$

let $W' = \psi^{-1}(X')$. codim $W' = 1$ so

$W = W' \oplus \mathbb{C}w$.

$$W \to X = (\mathbb{C} \xrightarrow{1} \mathbb{C}) \oplus (W' \to X')$$

2) Else $$W \xrightarrow{0} X \cong (\mathbb{C} \to 0)^{\oplus \dim W} \oplus (0 \to \mathbb{C})^{\oplus \dim X}.$$

Better still: "Only" nontriv s.e.s is

$$(*) \quad 0 \to (0 \to \mathbb{C}) \hookrightarrow (\mathbb{C} \to \mathbb{C}) \twoheadrightarrow (\mathbb{C} \to 0) \to 0$$

i.e. if $0 \to B \to M \to A \to 0$ then $\exists\, P, Q, k$ s.t.

$$\cong \quad (0 \to P \to P \oplus Q \to Q \to 0) \oplus \left((*)^{\oplus k} \right)$$

Subex 3: $Q = \circlearrowleft \bullet \circlearrowright$ $\quad Q\text{-rep}_{fd} = \mathbb{C}\langle x, y \rangle_{fd}$

Simples, Indecomps are awful.

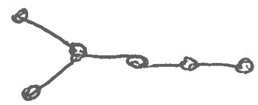
Qn: Which quivers have nice classification of simples / indecomp ??

<u>Gabriel's Theorem:</u> $Q\text{-rep}_{fd}$ has finitely many indecomposable objects

$\iff$ Q is an oriented Dynkin diagram of type ADE
(+ disjoint unions of these)

type A: $\circ - \circ - \circ - \circ - \circ \quad \overset{\text{oriented}}{\rightsquigarrow} \quad \circ \to \circ \to \circ \to \circ \leftarrow \circ \to \circ$

type D:



$E_6 \; E_7 \; E_8$:

Moreover,

$$\begin{array}{ccc} \{\text{indecomp}\}/\cong & \longleftrightarrow & \{\text{positive roots of root system}\} \\ \cup & & \cup \\ \{\text{simples}\}/\cong & \longleftrightarrow & \{\text{simple roots}\} \end{array}$$

<u>Ex:</u> $\circ - \circ$ Roots live in $\mathrm{Span}\overbrace{\{\alpha_1 = x_1 - x_2,\ \alpha_2 = x_2 - x_3\}}^{\text{simple roots}} \subset \mathbb{C}^3$

Pos roots $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2 = x_1 - x_3\}$

For either orientation send $\overset{W}{\circ} - \overset{X}{\circ}$ to $(\dim W)\alpha_1 + (\dim X)\alpha_2$

$\overset{\mathbb{C}}{\circ} - \overset{0}{\circ} \longmapsto \alpha_1$

$\overset{0}{\circ} - \overset{\mathbb{C}}{\circ} \longmapsto \alpha_2$

$\overset{\mathbb{C}}{\circ} - \overset{\mathbb{C}}{\circ} \longmapsto \alpha_1 + \alpha_2$

<u>Ex:</u> $\circ \to \circ \to \circ \leftarrow \circ \to \circ$ indecomp $\longleftrightarrow \{x_i - x_j\}_{1 \le i < j \le 6}$

$0 \to \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{1} \mathbb{C} \to 0 \longmapsto \alpha_2 + \alpha_3 + \alpha_4 = x_2 - x_5$

<u>"Gab. Thm" cont:</u> Q oriented affine Dynkin diagram $\rightsquigarrow$

$\exists$ 1-param family of indecomp whose dimension is the imaginary root.

(Moduli space is still tractable.)

<u>Ex:</u> $\tilde{A}_2$

$$\begin{tikzcd} \mathbb{C} \arrow[rr, "1"] & & \mathbb{C} \arrow[dl, "1"] \\ & \mathbb{C} \arrow[ul, "\lambda"] & \end{tikzcd}$$

(Similar to $M_{(1)}$ for $\mathbb{C}[x]$ $\bullet \circlearrowleft$)

<u>Ex:</u> $\tilde{D}_4$

$$\begin{tikzcd} \mathbb{C} \arrow[dr] & & \mathbb{C} \arrow[dl] \\ & \mathbb{C}^2 & \\ \mathbb{C} \arrow[ur] & & \mathbb{C} \arrow[ul] \end{tikzcd}$$

Why quivers? <u>Fun</u> + <u>interesting</u> playground, relatively easy. Nice functors to play with

<u>Warning:</u> Can also define <u>quivers w/ relations</u>

$$\begin{tikzcd} & \bullet \arrow[dr, "y"] & \\ \bullet \arrow[ur, "x"] \arrow[rr, "z"'] & & \bullet \arrow[loop below, "w"] \end{tikzcd} \qquad wz = yx$$

Any algebra is a quiver w/ relations! Linear category.

Many properties of Q-rep obviously won't generalize to quiver w/ relns.

<u>Note:</u> Quivers are a special case of R-Mod, see C-B Notes.