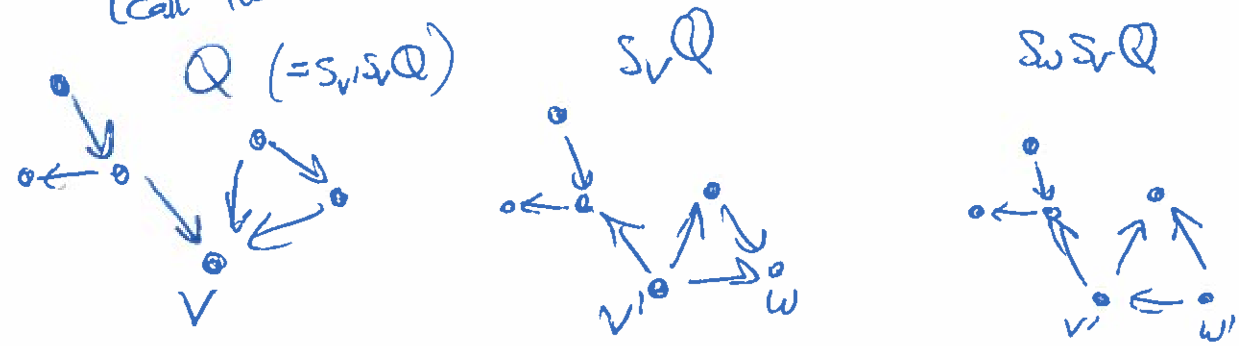
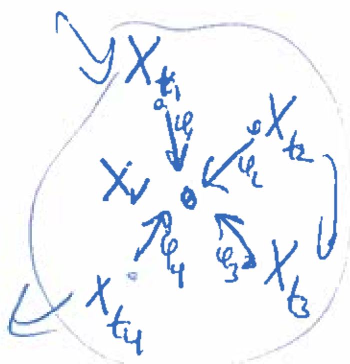
Reflection Functors

Def: Let Q be a quiver and $v \in V$ a source / ~~or~~ sink.
The v-reflected quiver S_vQ is obtained by reversing the arrows at v (call the vertex v' now) replacing it with a sink / source.



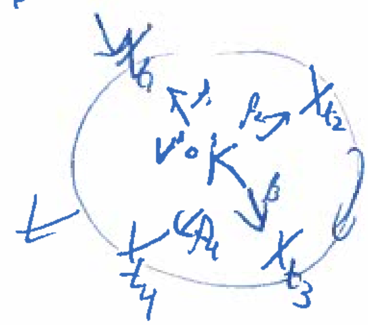
When v is a sink / source there's a funct $Q\text{-rep} \xrightarrow{S_v} S_vQ\text{-rep}$ I call sink pop / source push.



$\oplus\varphi_i : \oplus X_{t_i} \to X_v$. Then $K = \operatorname{Ker}(\oplus\varphi_i) \subset \oplus X_{t_i}$

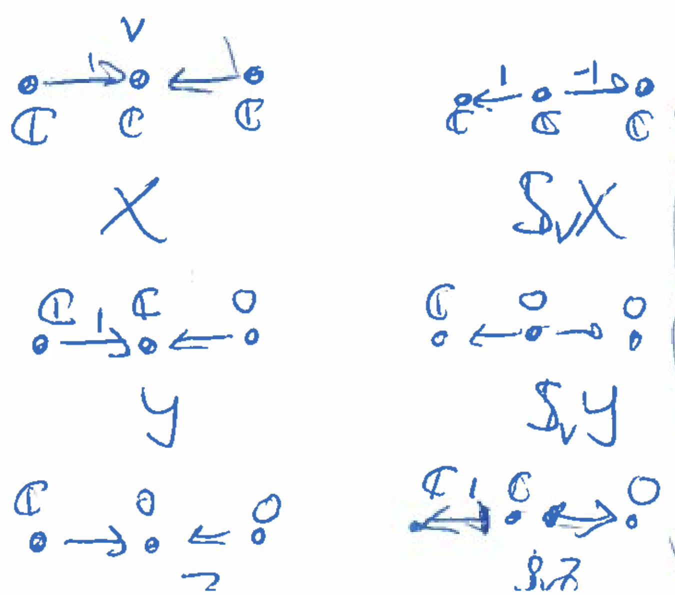
so $K \xrightarrow{p_i} X_{t_i}$

Let $S_vX =$



Sink pop

Ex:



looks like "involution"

$S_vW =$ 

but it kills the simple S_v.

Morphism:

$$\begin{tikzcd} \mathrm{Ker}_X \arrow[r, hook] \arrow[d] & \bigoplus X_{t_i} \arrow[r] \arrow[d] & X_v \arrow[d] \\ \mathrm{Ker}_W \arrow[r, hook] & \bigoplus W_{t_i} \arrow[r] & W_v \end{tikzcd}$$

Covariant

Things like kernels of operators are left exact.

If $X_v \hookrightarrow W_v$, $X_{t_i} \hookrightarrow W_{t_i}$ then by snake/5-lemma $\mathrm{Ker}_X \hookrightarrow \mathrm{Ker}_W$.

but it's definitely not right exact

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C} \arrow[d, "f"] & \mathbb{C} \arrow[l] \\ \mathbb{C} \arrow[r] & 0 & \mathbb{C} \arrow[l] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbb{C} & \mathbb{C} \arrow[l] \arrow[r] \arrow[d, "\exists f \text{ not surj}"] & \mathbb{C} \\ \mathbb{C} & \mathbb{C}^2 \arrow[l] \arrow[r] & \mathbb{C} \end{tikzcd}$$

It has right derived functors (compute w/ injectives)

I told you how to get projectives: P_v = span of paths from v

You might think J_v = span of paths to v is injective.

It's not ... it's not even a left module. It's a right module (precomposition)

Projective right module!

To make a left module take J_v^* i.e. $(J_v)_t^* := ((J_v)_t)^*$

~~dual of span of paths~~

Functions on paths t to v.

Q-rep $\overset{(\ )^*}{\rightleftarrows}$ rep-Q $\quad$ so $I_v := J_v^*$ is injective. Same size, easy to guess.

proj $\longleftrightarrow$ inj

$$\begin{array}{ccccc} & 1 & & 1 & \\ \bullet & \rightarrow & \bullet & \leftarrow & \bullet \\ \mathbb{C} & & \mathbb{C} & & \mathbb{C} \end{array}$$

is I_v.

$$\begin{array}{ccccc} & & & & t \\ \circ & \rightarrow & \circ & \leftarrow & \circ \\ 0 & & 0 & & \mathbb{C} \end{array}$$

is I_t.

$$R^i \mathcal{S}_v \left(0 \to \mathbb{C} \leftarrow 0 \right) = ?$$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] & 0 \arrow[l] \arrow[d] \\
\mathbb{C} \arrow[r, "1"] \arrow[d] & \mathbb{C} \arrow[d] & \mathbb{C} \arrow[l, "1"'] \arrow[d] \\
\mathbb{C} \arrow[r] \arrow[d] & 0 \arrow[d] & \mathbb{C} \arrow[l] \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

$I^\bullet \left(\begin{tikzcd} \mathbb{C} \arrow[r, "1"] \arrow[d] & \mathbb{C} \arrow[d] & \mathbb{C} \arrow[l, "1"'] \arrow[d] \\ \mathbb{C} \arrow[r] & 0 & \mathbb{C} \arrow[l] \end{tikzcd} \right) \overset{\mathcal{S}_v}{\rightsquigarrow} \left(\begin{tikzcd} \mathbb{C} \arrow[d] & \mathbb{C} \arrow[l, "1"'] \arrow[r, "1"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C} & \mathbb{C}^2 \arrow[l] \arrow[r] & \mathbb{C} \end{tikzcd} \right)$

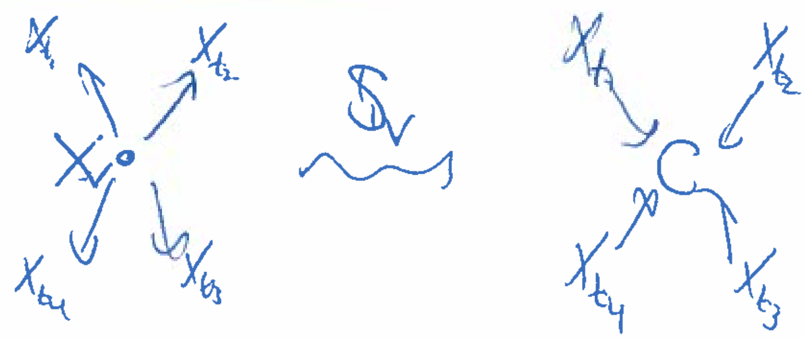
$h^0 = 0$

$h^1 = 0 \leftarrow \mathbb{C} \to 0$

so $R^0 \mathcal{S}_v(S_v) = 0$ but $R^1 \mathcal{S}_v(S_v) = S_{v'}$.

If you keep track of all $R^i \mathcal{S}_v$ then it is more like an involution!

Source push:



where $C = \mathrm{Coker}\left(X_v \longrightarrow \bigoplus X_{t_i} \right)$

exactness?

Fun exercises.