S.e.s. of complexes

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$$

Since Ker, Coker defined termwise, this is exact $\Leftrightarrow$ termwise exact.

$$\begin{tikzcd}
0 \arrow[r] & A^i \arrow[r, "f^i"] \arrow[d, "d_i"] & B^i \arrow[r, "g^i"] \arrow[d] & C^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] & B^{i+1} \arrow[r] & C^{i+1} \arrow[r] & 0
\end{tikzcd}$$

Warning: Termwise split $\Leftarrow$ split. $\not\Rightarrow$

Ex 1:

$$\begin{tikzcd}
& 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] \arrow[d] & 0 \arrow[r] & 0 & \\
& \vdots & & &
\end{tikzcd}$$

is termwise split but not split.

Fund Thm of Hom Alg: s.e.s of complex gives l.e.s of homology, i.e.

$$0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0 \text{ exact } \Rightarrow$$

$$\cdots \to h^{-1}(C) \xrightarrow{\delta^{-1}} h^0(A) \xrightarrow{f} h^0(B) \xrightarrow{g} h^0(C) \xrightarrow{\delta^0} h^1(A) \to h^1(B) \to \cdots \quad \text{exact}$$

Moreover, δ is functorial (for morphisms of s.e.s)

Ex 1:

$$0 \to 0 \to 0 \to \underset{h^0(C)}{\mathbb{C}} \underset{\sim}{\xrightarrow{\delta}} \underset{h^1(A)}{\mathbb{C}} \to 0 \to 0 \to \cdots$$

I'm not going to rehash the standard diagram chase proof.

Need to show

0) h^i is a functor

1) Can construct δ using Snake lemma

2) δ is functorial.

I'll try to give you new ways to think about it below instead.

For reason, I want to briefly remind step 1.

<u>Snake:</u>

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} d_A^i \arrow[r] \arrow[d] & \operatorname{Ker} d_B^i \arrow[r] \arrow[d] & \operatorname{Ker} d_C^i \arrow[r, dashed, "\tilde{\delta}"] \arrow[d] & {} \\
0 \arrow[r] & A^i \arrow[r] \arrow[d, "d_A^i"'] & B^i \arrow[r] \arrow[d] & C^i \arrow[r] \arrow[d, "d_C^i"] & 0 \\
0 \arrow[r] & A^{i+1} \arrow[r] \arrow[d] & B^{i+1} \arrow[r] \arrow[d] & C^{i+1} \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & \operatorname{Coker} d_A^i \arrow[r] & \operatorname{Coker} d_B^i \arrow[r] & \operatorname{Coker} d_C^i \arrow[r] & 0
\end{tikzcd}$$

Diagram chase gives $\tilde{\delta}$.

<u>Check:</u> $\tilde{\delta}$ kills image of d_C^{i-1}

so descends to $\underbrace{\operatorname{Ker} d^i / \operatorname{Im} d^{i-1}}_{h^i(C)"} \longrightarrow \operatorname{Coker} d_A^i$

<u>Check:</u> $\operatorname{Im} \hat{\delta}$ lives in $\operatorname{Ker}\left(d_A^{i+1}\big|_{\operatorname{Coker} d_A^i}\right)$

so lifts to map $h^i(C) \longrightarrow \operatorname{Ker} d^{i+1} / \operatorname{Im} d_A^i = h^{i+1}(A)$

(<u>Soon</u>: How to construct $\tilde{\delta}$ from univ. properties w/o diagram chase)

Let's make some s.e.s. of complexes. "The scoop on cones"

<u>Def:</u> $A^\bullet \xrightarrow{f} B^\bullet \rightsquigarrow \operatorname{Cone}(f)^\bullet :=$

$$\left(\begin{tikzcd}
{} \arrow[r] & B^{-1} \arrow[r, "d_B"] & \underline{B^0} \arrow[r, "d_B"] & B^1 \arrow[r] & {} \\
{} \arrow[ur] & \oplus & \oplus & \oplus & {} \\
{} \arrow[r] & A^0 \arrow[r, "-d_A"'] \arrow[uur, "f^0"] & \underline{A^1} \arrow[r, "-d_A"'] \arrow[uur, "f^1"] & A^2 \arrow[r] \arrow[uur] & {}
\end{tikzcd}\right)$$

Top row is a subcomplex, it's just $B^\bullet$

Bottom row is quotient complex, it's $A^\bullet[1]$

Def: $A^\bullet[1]$ is $\left(\cdots \to A^0 \xrightarrow{-d} \underline{A^1} \xrightarrow{-d} A^2 \to \cdots\right)$ Sign for reasons (wk 10)!

$f[1]$ is same map (no sign!)

Facts: 1) $0 \to B \xrightarrow{u} \mathrm{Cone}(f) \xrightarrow{v} A[1] \to 0$ is termwise split s.e.s.

2) $A \xrightarrow{f} B \xrightarrow{u} \mathrm{Cone}(f)$ is nonzero, but it is nullhomotopic.

Sim. $\mathrm{Cone}(f) \xrightarrow{v} A[1] \xrightarrow{f[1]} B[1]$ —//— .

3) Every termwise split s.e.s is a Cone!!!

If $0 \to B^\bullet \to C^\bullet \to D^\bullet \to 0$ termwise then 1) $C^i = B^i \oplus D^i$

2) top row is subcomplex

3) bot row is quot complex

4) Call diag terms f^i.

$$\begin{tikzcd}
B^{i-1} \arrow[r, "d_B"] & B^i \arrow[r, "d_B"] & B^{i+1} \\
\oplus & \oplus & \oplus \\
D^{i-1} \arrow[r, "d_D"'] \arrow[ur, "f"] & D^i \arrow[r, "d_D"'] \arrow[ur, "f"] & D^{i+1}
\end{tikzcd}$$

Then $f \circ d_D + d_B f = 0 \Rightarrow f$ "is" chain map $D[-1] \xrightarrow{\tilde{f}} B$

$\Rightarrow C = \mathrm{Cone}(f)$.

4) In l.e.s. for 1 we get

$$\begin{array}{c}
h^i(B) \to h^i(\mathrm{Cone}(f)) \to h^i(A[1]) \xrightarrow{\delta} h^{i+1}(B) \to \cdots \\
\qquad\qquad\qquad\qquad\qquad\quad \| \\
\qquad\qquad\qquad\qquad\qquad\quad h^{i+1}(A)
\end{array}$$

and $\delta = h^{i+1}(f)$!

Connecting maps for termwise split ses explained!!

Exercise.

Cones seem amazing but they are deeply problematic!

5) Cones are **not** canonical! Qn: What are the isos

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \mathrm{Cone}(f) \arrow[r] \arrow[d, "\varphi", "\wr"'] & A[1] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & \mathrm{Cone}(f) \arrow[r] & A[1] \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} B^i \oplus A^{i+1} \arrow[r, "{\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} id_B & h \\ 0 & id_A \end{pmatrix}}"'] & B^{i+1} \oplus A^{i+2} \arrow[d, "{\begin{pmatrix} id_B & h \\ 0 & id_A \end{pmatrix}}"] \\ B^i \oplus A^{i+1} \arrow[r, "{\begin{pmatrix} d & f \\ 0 & -d \end{pmatrix}}"'] & B^{i+1} \oplus A^{i+2} \end{tikzcd}$$

$$h : A^i \to B^{i-1}$$

For φ to be a chain map compute $d_B h = -h d_A$, i.e.

$$\boxed{h \text{ is a chain map } A \to B[-1]}$$

__Exercise:__ $\varphi \simeq id_{Cone} \iff h \simeq 0$ as chain maps $A \to B[-1]$

(i.e. $\exists$ homotopy $H : A \dashrightarrow B[-2]$ s.t. ..

So even in $K(\mathcal{A})$, a non-nullhomotopic map $A \to B[-1]$ gives rise to a nontrivial automorphism of $\mathrm{Cone}(f)$.

__Later:__ Cones of cones of cones ...