

Ses. of complexes

$$0 \rightarrow A^\circ \xrightarrow{f} B^\circ \xrightarrow{g} C^\circ \rightarrow 0$$

(1)

Since \ker, coker defined termwise, this is exact \Leftrightarrow termwise exact.

$$\begin{array}{ccccccc}
 0 & \rightarrow & A^i & \xrightarrow{f^i} & B^i & \xrightarrow{g^i} & C^i \rightarrow 0 \\
 & & \downarrow d_i & & \downarrow & & \downarrow \\
 0 & \rightarrow & A^{i+1} & \rightarrow & B^{i+1} & \rightarrow & C^{i+1} \rightarrow 0
 \end{array}$$

Warning: Termwise split \Leftarrow split.
~~*~~

Ex 1:

$$\begin{array}{ccccccc}
 0 & \rightarrow & \begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \vdots \end{array} & \rightarrow & \begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \vdots \end{array} & \rightarrow & \begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \vdots \end{array} \rightarrow 0 \\
 0 & \rightarrow & \begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \vdots \end{array} & \rightarrow & \begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \vdots \end{array} & \rightarrow & \begin{array}{c} \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \downarrow \\ \circ \\ \vdots \end{array} \rightarrow 0
 \end{array}$$

is termwise split but not split.

Fund ~~Thm~~ Thm of Hom Alg: s.es of complex gives lies of homology, i.e

$$0 \rightarrow A^\circ \xrightarrow{f} B^\circ \xrightarrow{g} C^\circ \rightarrow 0 \text{ exact} \Rightarrow$$

$$\dots \rightarrow h^1(C) \xrightarrow{\delta^1} h^0(A) \xrightarrow{f} h^0(B) \xrightarrow{g} h^0(C) \xrightarrow{\delta^0} h^1(A) \rightarrow h^1(B) \rightarrow \dots \text{ exact}$$

Moreover, δ is functorial (for morphisms of s.es)

Ex 1:

$$\begin{array}{ccccccc}
 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \mathbb{C} \xrightarrow{\delta} \mathbb{C} \rightarrow 0 \rightarrow 0 \rightarrow \dots \\
 & & & & & & h^1(C) & & h^1(A)
 \end{array}$$

I'm not going to rehash the standard diagram chase proof.

(2)

- Need to show
- 0) h^i is a functor
 - 1) Can construct \mathcal{S} using Snake lemma
 - 2) \mathcal{S} is functorial

I'll try to give you new ways to think about it below instead. For reason, I want to briefly remind step 1.

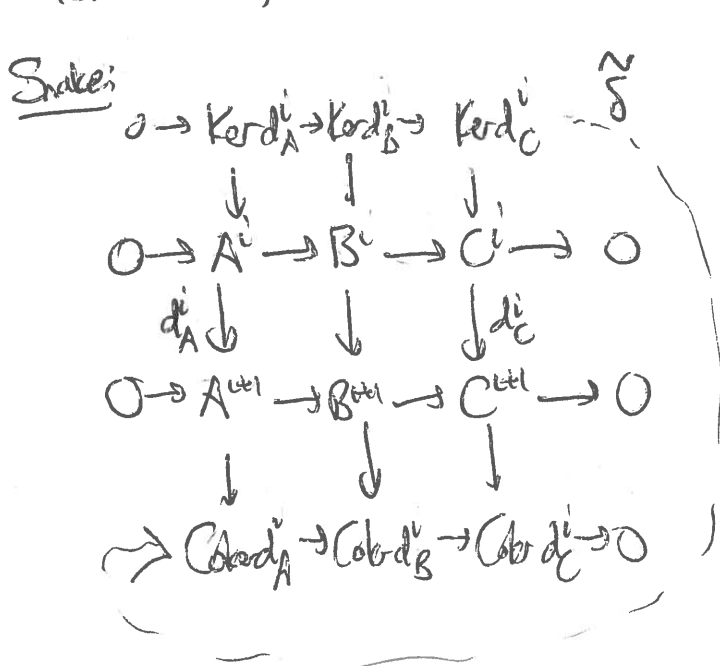


Diagram chase gives $\tilde{\mathcal{S}}$.

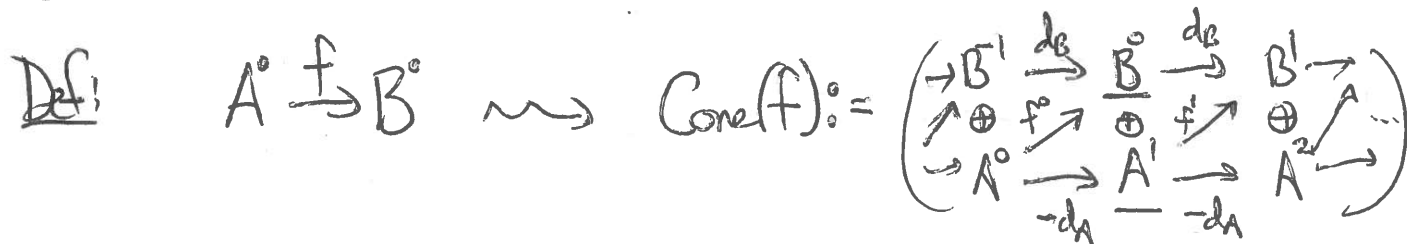
Check: $\tilde{\mathcal{S}}$ kills image of d_C^{i-1}
 so descends to $\text{Ker } d_C^i / \text{Im } d_C^{i-1} \rightarrow \text{Coker } d_A^i$
 $h^i(C)$

Check: $\text{Im } \tilde{\mathcal{S}}$ lies in $\text{Ker} \left(\begin{array}{c} d_A^{i+1} \\ A \end{array} \middle| \text{Coker } d_A^i \right)$
 so lifts to map $h^i(C) \rightarrow \text{Ker } d_A^{i+1} / \text{Im } d_A^i = h^{i+1}(A)$

Soon: How to construct $\tilde{\mathcal{S}}$ from our properties w/o diagram chase.

Let's make some seq. of complexes.

"The scoop on cones"



Top row is a subcomplex, it's just B^\bullet

Bottom row is quotient complex, it's A^\bullet

Def: $A[\mathbb{Z}]$ is $(\dots \rightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \rightarrow \dots)$
 $f[\mathbb{Z}]$ is same map (no signs!)

Sign for $\textcircled{3}$
reason (wk 10!)

Facts: 1) $0 \rightarrow B \xrightarrow{u} \text{Cone}(f) \xrightarrow{v} A[\mathbb{Z}] \rightarrow 0$ is terminating split s.e.s.

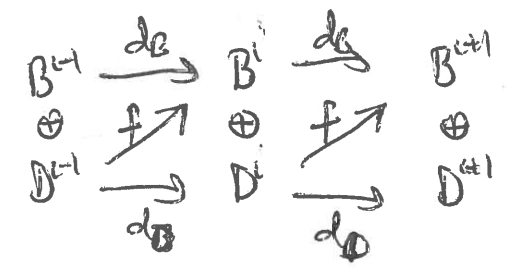
2) $A \xrightarrow{f} B \xrightarrow{u} \text{Cone}(f)$ is nonzero, but f is nullhomotopic.

Sim. $\text{Cone}(f) \xrightarrow{v} A[\mathbb{Z}] \xrightarrow{f[\mathbb{Z}]} B[\mathbb{Z}]$ ~~_____~~

3) Every terminating split s.e.s. is a Cone $\textcircled{1}$

If $0 \rightarrow B^i \rightarrow C^i \rightarrow D^i \rightarrow 0$

terminating then 1) $C^i = B^i \oplus D^i$



- 2) top row is subcomplex
- 3) bot row is quot complex

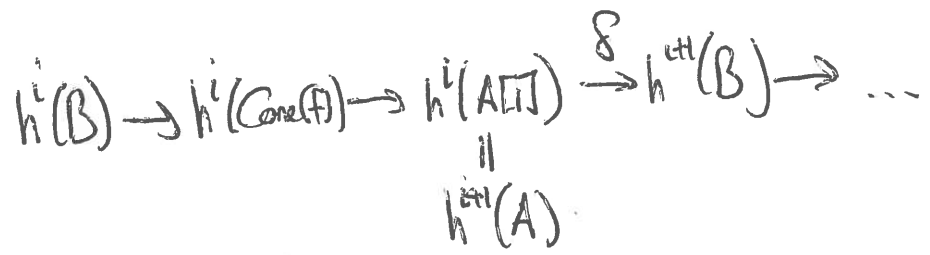
4) Call diag terms f^i .

Then $f \circ d_B + d_D \circ f = 0 \Rightarrow f$ "is" chain map

$D[\mathbb{Z}] \xrightarrow{\tilde{f}} B$

$\Rightarrow C = \text{Cone}(f)$.

4) In l.e.s. for $\textcircled{1}$ we get



and $\delta = \circlearrowleft h^{i+1}(f)$ $\textcircled{1}$

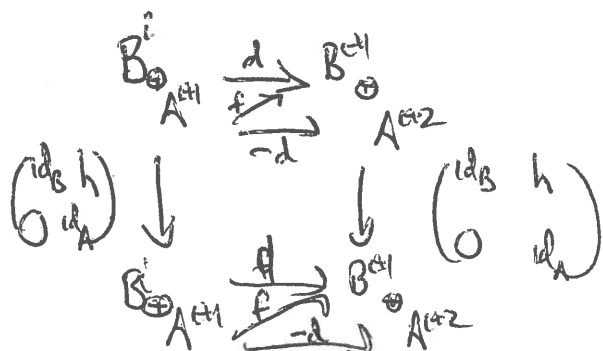
Connecting maps for terminating split s.e.s. explained $\textcircled{1}$

Exercise.

Cores seem amazing but they are deeply problematic!

④

5) Cores are not canonical! Qn: What are the isoms



$$\begin{array}{ccccccc}
 0 & \rightarrow & B & \rightarrow & \text{Core}(F) & \rightarrow & A[1] \rightarrow 0 \\
 & & \parallel & & \psi \downarrow & & \downarrow \\
 0 & \rightarrow & B & \rightarrow & \text{Core}(F) & \rightarrow & A[1] \rightarrow 0
 \end{array}$$

$$h: A^i \rightarrow B^{i-1}$$

For ψ to be a chain map compute $d_B h = -h d_A$, i.e.

$$h \text{ is a chain map } A \rightarrow B[-1]$$

Exercise: $\psi \simeq \text{id}_{\text{Core}}$ $\Leftrightarrow h \simeq 0$ as chain maps $A \rightarrow B[-1]$

(i.e. \exists homotopy $H: A \cdots \rightarrow B[-2]$ s.t.

So even in $\mathbb{K}(A)$, a non-nullhomotopic map $A \rightarrow B[-1]$ gives rise to a nontrivial automorphism of $\text{Core}(F)$.

Later: Cores of cores of cores...