**Spectral Sequences**

By the time we're done, we'll have seen many perspectives, but it'll try to start easy.

**Def.** A bicomplex in $A$ is:

$A_{\cdots} \rightarrow A_{\cdots} \rightarrow A_{\cdots} \rightarrow A_{\cdots}$

subject to:

1. $d_h^2 = 0$ (rows are complex)
2. $d_v^2 = 0$ (cols are complex)
3. $d_h d_v + d_v d_h = 0$ (total is a complex - soon)

Equivalently: a graded $\wedge^\bullet(\mathbb{Z}^2) = \mathbb{Z} d_1 \wedge d_j / d_1 d_2 + d_2 d_1 = d_1 d_2 = d_2 d_1 = 0$.

$a_{\cdots}$

3 straightforward ways to bicomplexes:

1. Forget $d_h$, get a bunch of row complexes. $h_{\text{horiz}}(A^\bullet) = \ker d_h^\bullet / \text{Im } d_h^\bullet$
2. Forget $d_v$, get column complexes.
3. **Total complex** $\text{Tot } A^\bullet$ in degree $k$ is $\bigoplus_{i+j=k} A_{ij}$

and $d_{\text{tot}} = d_h + d_v$, so $A_{ij} \rightarrow A_{ij}^{k+1}$

$d_{\text{tot}}^2 = d_h^2 + (d_h d_v + d_v d_h) d_v^2 = 0$ as desired.

**Ex:**

\[
\begin{array}{c|c|c|c}
\text{horiz} & \text{vert} & \text{total} \\
\hline
0 & A & 0 & 0 \\
0 & 0 & 0 & 0 \\
A & 0 & C & C \\
0 & C & 0 & 0 \\
C & 0 & 0 & C \\
\end{array}
\]

**HUGE WARNING:** Also have $\text{Tot}^\mu(A)^k = \bigoplus_{i+j=k} A_{ij}$ and can be very different!!

**Ex:**

\[
\begin{array}{c|c|c|c}
\text{horiz} & \text{vert} & \text{total} \\
\hline
0 & A & 0 & 0 \\
0 & 0 & 0 & 0 \\
C & 0 & C & C \\
0 & C & 0 & 0 \\
C & 0 & 0 & C \\
\end{array}
\]

Both $\text{Tot}^0$ and $\text{Tot}^\mu$ have only degree 0 and 1.

In $\text{Tot}^0$, $d_{\text{tot}}$ is injective. In $\text{Tot}^\mu$, $d_{\text{tot}}$ is not.
There are many, many interesting boundedness conditions, and for many of these, \( \text{Tot}^0 = \text{Tot}^\Pi \).

Not worth keeping them straight. But difference between \( \text{Tot}^0 \) and \( \text{Tot}^\Pi \) is a big deal!

May useful examples of bicomplexes + their characteristics:

1. \( A^0 \xrightarrow{f} B^0 \) chain map of ordinary complexes
   \[ \text{fd}_A = \text{df} \]

2. \( A^1 \xrightarrow{f^1} B^1 \) \( -dA \xrightarrow{f} B^0 \)
   \[ \text{wd signs, d} \circ \text{d} = 0 \]
   \[ \text{but redefining d}, \text{get a bicomplex} \]

Sign idea: \( A \) is now shifted in \( \text{Tot} \) banded total degree,

\[ A^0 \text{ in degree } -1. \]

\[ \text{AE}[] \] is the complex appearing.

When you shift a complex, you should also negate all differentials.

\[ \text{Tot}^0 (\text{bicomplex}) \equiv \text{Cone}(f) = \]
\[ \begin{array}{c}
  B^0 \xrightarrow{f} B^0 \\
  \text{fd}_A = \text{df} \\
 \end{array} \]

This has \( B^0 \) as a subcomplex and \( \text{AE}[] \) as a quotient complex.

More on cone later!

What about cohomology?

Hirze:\n
\[ \ker f' \xrightarrow{i} \ker f \]
\[ \ker f'' \xrightarrow{i} \ker f' \]
\[ \ker f'' \xrightarrow{i} \ker f' \]

\[ h^0(A) \xrightarrow{f} h^0(B) \]
\[ h^1(A) \xrightarrow{f} h^1(B) \]

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\[ h^0 \text{ inherits } \text{d} \]
\[ h^1 \text{ inherits } \text{d} \]

\[ \text{easy exercise} \]
There is a relationship between $h_{\text{vert}}(h_{\text{horz}}(\cdot))$ and $h_{\text{vert}}(\cdot)$ but it is complicated: "spectral sequence".

Claim: $h^k(\text{Cone}(f)) \Rightarrow \text{admits a b.s.a.}$

Consider $h^0(\text{Cone}(f))$ total degree $0$

$$
\quad \xymatrix{ 0 \ar[r] & \text{Coker } f^k \ar[r] & h^k(\text{Cone}(f)) \ar[r] & \ker f^{k+1} \ar[r] & 0 }
$$

Consider $h^0(\text{Cone}(f))$. $\ker f^0$ is $\begin{bmatrix} b \end{bmatrix}$, $\begin{bmatrix} a \end{bmatrix}$. st. $\partial b = -fa$

Then $\alpha \in \ker f^0 \Rightarrow \alpha \in h^1(A)$. $f^k(a) = \frac{fa}{h^k(B)}$, but $fa = db$ so $\alpha = 0$.

If $\alpha \in h^1(A)$ is s.a., $\exists \alpha \in A^0$, $-d\alpha = \alpha \Rightarrow \begin{bmatrix} 0 \end{bmatrix}$ and $\begin{bmatrix} a \end{bmatrix}$ agree in $h^0(\text{Cone}(f))$

Thus $\text{Tor}$ is well defined $h_\text{vert} h_{\text{horz}} \Rightarrow (\text{Im} \partial_B + \text{Im} F | \text{Ker} h_{\text{horz}})$, exercise...

Claim: There is a b.s.a.

$$
\quad \xymatrix{ h^k(\text{Cone}(f)) \ar[r] & h^k(\text{Cone}(f)) \ar[r] & h^k(\text{Cone}(f)) \ar[r] & h^k(\text{Cone}(f)) \ar[r] & \cdots }
$$

How to understand this relationship?

Bi-complexes:

A complex of complexes $B^1 \xrightarrow{d_1} B^2 \xrightarrow{d_2} \cdots$ give rise to a bi-complex $B^i_j$,

where $d_1 \circ d_2 = 0$.

How to understand this relationship?

Relationship at cohomology even worse.

\[d' \circ d = 0 \text{ sign, no death complex}\]

$\partial$-death complex
Subex. Space \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) s.c.s. of complexes

\[
\begin{align*}
\text{h}_{\text{hom}} &= 0 \quad \Rightarrow \quad \text{h}_{\text{tot}} = 0. \\
\text{h}_{\text{tot}}(\text{h}_{\text{hom}}) &= \begin{bmatrix}
q^0 & q^1 & q^2 \\
\ell & \ell & \ell \\
\ell & \ell & \ell 
\end{bmatrix} \\
\text{Y/C of les} &= \begin{bmatrix}
\ell & \ell & \ell \\
\ell & \ell & \ell \\
\ell & \ell & \ell 
\end{bmatrix} \rightarrow \begin{bmatrix}
\ell & \ell & \ell \\
\ell & \ell & \ell \\
\ell & \ell & \ell 
\end{bmatrix}
\end{align*}
\]

Again, some kind of new map of degree \((-d+1)\)...

3) \( \text{Tor}^k(M,N) \) M right R mod. \( N \) left R mod.

\( P^0 \rightarrow M \) right proj resn \( \phi \rightarrow N \) left proj resn.

\( P^0 \rightarrow M \rightarrow N \rightarrow P^0 \rightarrow N \)

\( P^0 \rightarrow P^0 \rightarrow P^0 \rightarrow M \rightarrow N \)

\[
\begin{align*}
\text{h}_{\text{hom}} &= \begin{bmatrix}
\text{Tor}_2^2(M) & \text{Tor}_1^2(M) \\
0 & 0
\end{bmatrix} \\
\text{hom} &= \begin{bmatrix}
0 & 0 \\
\text{Tor}_2(M) & \text{Tor}_1(M)
\end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix}
\phi \\
\phi \\
\phi
\end{bmatrix}
\]

need signs!

Consider augmented bicomplex.
All rows exact except row \( 0 \) -
they compute \( \text{Tor}^k(M,Q) = 0 \) since \( Q \) is project.
All cols exact except col \( 0 \), \( \text{Tor}^k(N) \)

\[
\begin{align*}
\text{h}_{\text{tot}} &= \begin{bmatrix}
0 & 0 \\
\text{Tor}_1(N) & \text{Tor}_0(N)
\end{bmatrix} \\
\text{Tor}_0(M,N) &= \text{Tor}_1(M,N)
\end{align*}
\]

Given well-defined map \( \text{Tor}^2 \rightarrow \text{Tor}^2 \)

\[
\begin{align*}
\text{indep of choices} \quad \text{Zig-Zag} / \text{Stasheff}
\end{align*}
\]
Ready for the general theory yet?
Take a bicomplex $A^{\ddagger}$, then we get:

1. $E_0^{\ddagger}$ is $h_{\text{vert}}(A^{\ddagger})$, has differential induced from above, degree $(0, 1)$
2. $E_1^{\ddagger}$ is $h_{\text{horiz}}(A^{\ddagger})$, has differential induced from above, degree $(1, 0)$
3. $E_2^{\ddagger}$ is $h_{\text{horiz}}(h_{\text{vert}}(A^{\ddagger}))$, has induced differential of degree $(2, -1)$

\[
\begin{align*}
& x \rightarrow w \\
& \text{descends to well defined map, independent of choice.}
\end{align*}
\]

$\text{vert} \Rightarrow A^{pq} \rightarrow h_{\text{horiz}} \Rightarrow A^{pq}$,

\[
\begin{align*}
\text{take \: chomotpy} \\
\text{electron}
\end{align*}
\]

The collection of all these pages and differentials is the spectral sequence attached to

the bicomplex $A^{\ddagger}$, having chain vert then horiz.

If you do homs the vert get differentials of degree $(1, 0)$, $(0, 1)$, $(1, 0)$, etc...
If $A^{\ddagger}$ has certain bounded ness conditions, $d_{k=0}$ with to and from $A^{pq}$ for fixed $p, q$

Thus implies $E_{k}^{pq} = E_{k}^{pq} = E_{k}^{pq}$ \(\forall k \geq K\), call this $E_{\infty}^{pq}$.
What is the relationship between $E_0^{\ddagger}$ and $H_{\text{horiz}}(A^{\ddagger})$?
We say $E_{pq}$ converges to $h^0$ if $h^k$ has a filtration w/ subquotients.

(assuming bilinear conditions)

We say $E_{pq}$ converges to $h^0$ if $h^k$ has a filtration w/ subquotients

$$E_{pq} \xrightarrow{c_{j,k}} h_j \xrightarrow{c_{j,k}} h_k$$

$s.t.$ $F^k_i / F^k_{i-1}$ is finite, w. bilinear conditions

(If not finite, also have to ask if $\bigcap F^k_i = 0$ and $\bigcup F^k_i = h^k$, etc, so in other notion of convergence when these fail, will ignore this shit.)

(assuming bilinear conditions) $E_{pq}$ of a double complex converges to $h^0(Tot^a)$.

**Exs:**

1. $A^{\bullet\bullet} = \begin{pmatrix} C & C \\ C & C \\ C & C \end{pmatrix}$

**Important:** Can NOT use this to compute $h^0(Tot\ A)$ recursively — only get to subobjects in a filtration, not the object itself.

If in a semisimple category like Vect, it is good enough though (Bredon (sheaves needed))

If exact, even better result.

Examples:

1. $A^{\bullet\bullet}$

**Important:** Can NOT use this to compute $h^0(Tot\ A)$ recursively — only get to subobjects in a filtration, not the object itself.

If in a semisimple category like Vect, it is good enough though (Bredon (sheaves needed))

If exact, even better result.
Some complex where $h^{\text{max}}$ concentrates in one column, but in a row

Filtration on $h^{\text{max}}$ has only one term so $h^{k}_{\text{top}} = h^{k}(B')$

For large $E_0$, $E_1$, $E_2 = E_{\text{top}}$

If at any point $E^p_k$ vanishes like a checkerboard then $E^p_k = E^p_{\text{top}}$

Partly miracle!

Utility/Example 1: Leray spectral sequence $F \to E \to B$ fiber bundle, $\pi_1(B) = 0$

$E^2_{pq} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E)$

Where $E_0$, $E_1$, Can fill the simplicial chain complex, but more natural to take chain $\Rightarrow$ step $\Rightarrow$ hole, $\partial_i$, etc. difficult though!!
Hyper-derived functors

How do derived functors work?
- Take $\mathcal{A}$
- Form a complex: $0 \to M \to 0$
- Find a quasi-isomorphism $p^2 \to p^1 \to M \to 0$
- Apply functor to $p^2$ instead
- Take homology

What if you replaced $M$ with $M'$?
- Start with a $M'$ complex $(M'^n)$
- Find a quasi-isomorphism $p^2 \to p^1 \to M'^{n-1} \to M'$
- Continue

This is called the hyper-derived functor $L^m F$. I'm happy if you just call it a derived functor.

$L^0 F : A(A) \to B$

But there's another thing you can do:

$M^\bullet \in A(A) \to B$

$L^m F(M^\bullet)$

Relationship: this is exactly the kind of situation when you get a spectral sequence.

For a third

$A(A) \to B$

$L^m F(M^\bullet)$

Definition: A is exact at $M^n$. A Cartan-Eilenberg resol of $M$ is a double complex $p^\bullet$.

$s$.

In particular, all the things on the top, $p^0$ is very easy.

(Just remember to choose the kernel? Very easy.)

Only need to assume b), d) and not follow.

This goes with the $\mathcal{A}$ Hand. There are three.

Moreover, maps of complexes $M^\bullet \to N^\bullet$ induce maps of CE resol, $p^\bullet \to q^\bullet$ up to homotopy of bicomplexes (see book)

Moreover, $Tot p^\bullet \to M$ is a quasi-iso homotopy of bicomplex is homotopy of $Tot$
3. Gorth spectral seq

$E_{2}^p = L^p_G(E^p(M)) \Rightarrow L^{p+q}(E^0_f)(M)$

**Proof:**

$P^p \to M$, apply $F$ get $(FP^p) \to FM$

$\text{nor-bk}$ $L^q_G(FP^p) \text{ via } (FP^p \to FM) \xrightarrow{G} \text{ get } G^p \to GM$

S.S. gives $E_{2}^p = L^{p+q}(G(FP^p)) \Rightarrow L^{p+q}G(FP^p)$

Now $L^q_G(FP^p) = 0$ for $q > 0$, since each $FP^p$ is $G$-acyclic. Consequently, in $q = 0$,

where it is just $h^p(G(FP^p)) = L^p(Gf)(M)$. So it vanishes immediately at $E_2$,

and $L^{p+q}(G(FP^p)) = L^{p+q}(Gf)(M)$.

Now $E_{2}^p = \text{Dev} L^p_G(h^q(FP^p)) = L^p_G(L^q(M))$, as desired. \(\checkmark\)

Many more examples of this.

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Kharom-style

Consider a complex in $\text{Vechik}$. Then an upgraded model $\text{K}^0(\text{Vechik})$.

The $\text{alg}$ is free, $\text{alg}$ is not.

Indecomposable: projections and staircases.

Need only see how S.S. behaves for each staircase.

Odd length:

$E_2 = 0 = E_0 = E_0$
Each Erc:II helicase contributes precisely one dimer, i.e., passage from Ecoz to Ecoz.

just kill copies of one helicase at a time eventually.

(No worries about fitting here — in fact, every split)

Fitted complexes + convolution
What about a convolution w/ 3 pieces? \( f = d_2 \cdot 1 \) \( g = d_{1-o} \) \( h = d_{2-o} \)

\[
\begin{align*}
\mathbb{C}^0 & \xrightarrow{d_2} \mathbb{C}^1 \xrightarrow{d_1} \mathbb{C}^2 \\
\mathbb{B}^0 & \xrightarrow{d_2} \mathbb{B}^1 \xrightarrow{d_1} \mathbb{B}^2 \\
\mathbb{A}^0 & \xrightarrow{d_2} \mathbb{A}^1 \\
\end{align*}
\]

so a convolution commutes

\[ f \circ A^0 \rightarrow B^0 \rightarrow C^0 \]

but we fixed chain to be nullhomotopic!!

Conditions w/ 3 pieces where \( d_{2-o} = 0 \), i.e. \( h = 0 \), i.e. actual complex of complexes. This would be the Tot of

\[ \text{Tot} \mathbb{C}^0 \]

but with \( h \neq 0 \), don't get Tot of any bicomplex!!

Condition 2:
A convolution is purely degree-1 if \( I = \mathbb{Z} \) and \( d_j = 0 \) unless \( j = i+1 \) or \( i \)

Easy check:

purely degree-1 convolution \[ \hookrightarrow \] Tot of bicomplexes?

More relatable:
If \( A^0 \rightarrow B^0 \rightarrow C^0 \) w/ \( g^j = 0 \) then get induced maps \( \text{Core}(f) \rightarrow \text{Core}(g) \)

Taking care of these, get the full convolution. (If one meg, they're notifiable isomorph.)

The map \( A^0/\mathbb{Z} \rightarrow \text{Core}(g) \) only hits the \( B \) layer / the map \( \text{Core}(f) \rightarrow C^0 \) is zero on the \( A \) level
But there are maps \( A^0 \rightarrow B^0 \) and maps \( \text{Core}(f) \rightarrow C^0 \) which aren't zero on both \( A \) and \( B \), and these give non-deg-1 convolutions when one is taken.

Restricting to \( B^0 \rightarrow C^0 \) get g w/ \( -g^j = 0 \) nullhomotopy h.

Anyway, there is also the spectral sequence attached to a filtered complex:

\[
E_0^{k-i} = \bigoplus_{n \in \mathbb{Z}} \text{Coker}(d^1_{k+1-n})/\text{Im}(d^1_{k-n})
\]

\[
d^1 \colon h^k(F_G/F_{\geq 1}G) \rightarrow h^k(F_{	ext{/top}}/F_{\geq 2}G)
\]

\[
d^2 \text{; worse...}
\]
But the part is that a convolution is "like" a (co)chain with hyper-differential already!

\[ A^2 \rightarrow B^2 \rightarrow C^2 \rightarrow \cdots \]

\[ A^0 \rightarrow B^0 \rightarrow C^0 \]

Details: swept blissfully away.

Then \((E^{pq}, d_k) \Rightarrow H^*(C)\) under some bdd's and things.

But not every (co)chain is degenerate (split/filled), let alone a Cartan complex!

May carefree. I mean apply a functor to a (co)chain complex yields a (co)chain (complex).

Then: Even better, \((F(M), \delta)\) (homology thesis) \(\exists\) spectral sequence \(\Rightarrow\)

\[ E_p^{pq} = L^p G(P_i) \Rightarrow L^p h_* (M) \]

Then: Observe that a (co)functor \(M^i = (\cdots - M^{i-1} - M^i - M^{i+1} - \cdots)\) has a (co)kernel

with \( \delta^i(M) = (0 \cdots 0 - M^i - 0 \cdots 0) \Rightarrow M^i \]

Consequently \( E_p^{pq} = L^p G(M^i) \Rightarrow L^p h_* (M) \) (check indexing)

Again: Key part: aside of homology, spectral sequence are mostly useless

except when they degenerate at \(E_2\) (completely hyper-difficult is hard)

\( E_2 \Rightarrow E_\infty \) complex of sheaves

Ex! I have recent paper where I compute \( h_0^H (C) \)

by finding a domain \( C^0 \) as a convolution of simpler terms, which can be concretely

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\( H^H (C) \) (use case) is in even degrees only \( \Rightarrow E_2 = E_\infty \) so

of splinters, etc. etc.

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of splinters, etc. etc.
Roberts philosophy + Gaussian elimination

\[ \rightarrow A \xrightarrow{d} B \xrightarrow{e} \overline{B} \xrightarrow{f} \overline{C} \xrightarrow{g} \overline{D} \xrightarrow{h} \overline{E} \]

\[ \begin{bmatrix} a & b \end{bmatrix} \]

Every complex \( A \rightarrow C \rightarrow D \rightarrow E \rightarrow \overline{C} \rightarrow \overline{E} \) is isometry equivalent. Draw the maps.

Exercise: They are:

\[ \begin{array}{c}
\text{Exist} & \quad \text{contract out} & \quad \text{at opt =} \\
C & \rightarrow & \overline{C} \\
C & \rightarrow & C
\end{array} \]

This is really effective tool. Encapsulation. I use it often.

Note: In ch. (Next) or any sense, the cary cannot map in many so can repeatedly Gaussian elimination

with all elements on zero and only the cohomology remains. But this is not just chopping off iron,

since you can create new iron, as in the example above.

Since you have a convolution / split-point, consider. When carry Gaussian elimination, the terms which cancel

my le is different filtered degree. One algorithm:

Gaussianeliminate unit and later first (diff on),

then deal then diff etc.

E is not the original, but th new differential after GE.

Exercise:

\[ C \rightarrow C \]

is filtration of alone except by

\[ C \rightarrow C \]

Applying, first GE is just taking hext.

Get

\[ C \xrightarrow{\alpha_2} \overline{C} \]

Idea/part: The different on the Ek page is a check that is eliminated by the \( i^{th} \text{col} (C+\overline{C}) \).

This is three, depends on previous element. Original complex atom had \( d_{20} = 0 \).

When you're finished all GE, end up with \( h^*(\text{complex}) = h^*(\text{to } E_0) = E_0. \)