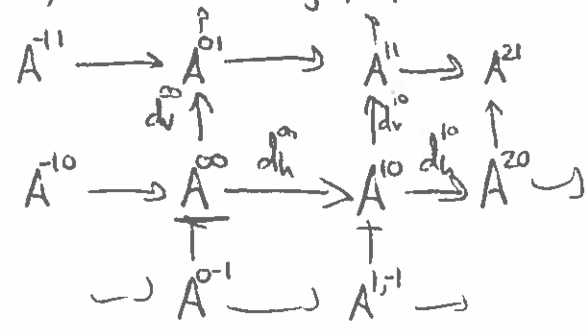
Spectral Sequences By the time we're done, we'll have seen many perspectives, but I'll try to start easy.

Def: A **bicomplex** in $\mathcal{A}$ is:



s.t.
- $d_h^2=0$ (rows = complexes)
- $d_v^2=0$ (cols = complexes)
- $d_v d_h + d_h d_v = 0$ (total is a complex – soon)

Equivalently: a graded $\Lambda^*(\mathbb{Z}^2) = \mathbb{Z}[d_1,d_2] / d_1 \wedge d_1 = d_2 \wedge d_2 = d_1 \wedge d_2 + d_2 \wedge d_1 = 0$ – module in $\mathcal{A}$.

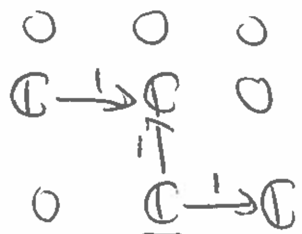
3 straightforward ways to bicomplex ⇝ complex / 3 kinds of cohomology.

1. forgetting d_v, get a bunch of row complexes. $h^{ij}_{horiz}(A^{\bullet\bullet}) = \operatorname{Ker} d_h^{ij} / \operatorname{Im} d_h^{i-1,j}$
2. forget d_h, get column complexes.
3. **Total complex** $\operatorname{Tot}^{\oplus} A^{\bullet\bullet}$ in degree k is $\bigoplus_{i+j=k} A^{ij}$ $\qquad h^k_{tot} = \operatorname{Ker} d^k_{tot} / \operatorname{Im} d^{k-1}_{tot}$

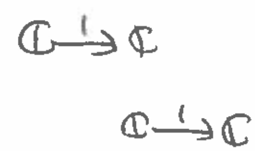
   and $d_{tot} = d_h + d_v$, sends $A^{ij} \longrightarrow A^{i+1,j} \oplus A^{i,j+1}$

$$d_{tot}^2 = d_h^2 + (d_h d_v + d_v d_h) + d_v^2 = 0 \text{ as desired.}$$

Ex:

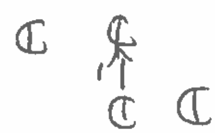


horiz



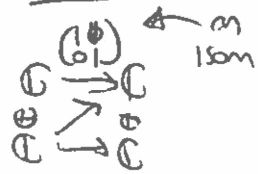
$h^{ij}_{horiz} = 0 \ \forall i,j$

vert



$h^{ij}_{vert} = \begin{cases} \mathbb{C} & (1,0) \\ & (-1,1) \\ 0 & \text{else} \end{cases}$

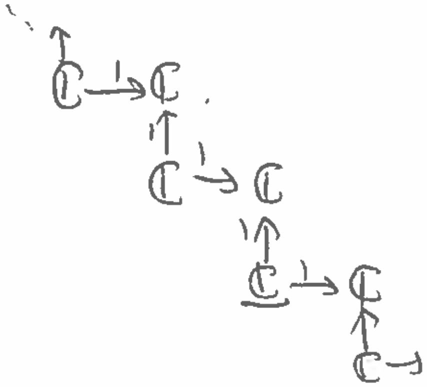
total



$h^k_{tot} = 0 \ \forall k$.

HUGE WARNING: Also have $\operatorname{Tot}^{\Pi}(A)^k = \prod_{i+j=k} A^{ij}$ and can be very different!!

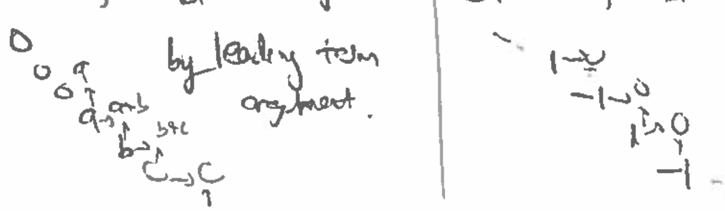
Ex:



both $\operatorname{Tot}^{\oplus}$ and Tot^{Π} have only degree 0 and 1

In $\operatorname{Tot}^{\oplus}$, d_{tot} is injective by leading term argument.

In Tot^{Π}, d_{tot} is not



There are many, many interesting boundedness conditions, and for many of these, $\mathrm{Tot}^{\oplus} = \mathrm{Tot}^{\Pi}$

Not worth keeping them straight. But difference b/w $\mathrm{Tot}^{\oplus}$ and Tot^{Π} is a big deal!!

<u>Many useful examples of bicomplexes</u> + their cohomologies

① $A^\bullet \xrightarrow{f^\bullet} B^\bullet$ chain map of ordinary complexes $\quad f d_A = d_B f$

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
0 & & A^1 & \xrightarrow{f^1} & B^1 & & 0 \\
 & & -d_A\uparrow & & \uparrow d_B & & \\
0 & & A^0 & \xrightarrow{f^0} & \underline{B^0} & & 0 \\
 & & -d_A\uparrow & & \uparrow d_B & & \\
0 & & A^{-1} & \xrightarrow{f} & B^{-1} & & 0 \\
 & & \uparrow & & \uparrow & &
\end{array}$$

← w/o signs, $d_v d_h - d_h d_v = 0$ ✗

but negating d_A, get a bicomplex

[<u>Sign idea</u>: A is now shifted in Tot homological degree, $A[1]$ is the complex appearing A^0 in degree -1. When you shift a complex, you should also negate all differentials!

$$\mathrm{Tot}^{\oplus}(\mathrm{bicomp}(A^\bullet \xrightarrow{f} B^\bullet)) \equiv \mathrm{Cone}(f) = \quad \cdots \begin{array}{ccccc} B^{-1} & \xrightarrow{d_B} & B^0 & \xrightarrow{d_B} & B^1 \\ \oplus & \nearrow f & \oplus & \nearrow f & \oplus \\ A^0 & \xrightarrow{-d_A} & \underline{A^1} & \xrightarrow{-d_A} & A^2 \end{array} \cdots$$

this has $B^\bullet$ as a subcomplex and $A[1]$ as a quotient complex
↖ has hom. shift + signed differentials.

More on cones later!

~~② More generally~~ What about cohomology?

h_{horiz}:

$$\begin{array}{ccc}
\operatorname{Ker} f^1 & & \operatorname{Coker} f^1 \\
\uparrow & & \uparrow \\
\operatorname{Ker} f^0 & & \underline{\operatorname{Coker} f^0} \\
\uparrow & & \uparrow \\
\operatorname{Ker} f^{-1} & & \operatorname{Coker} f^{-1}
\end{array}$$

h_{vert}

$$\begin{array}{c}
h^1(A) \xrightarrow{f} h^1(B) \\
\\
h^0(A) \xrightarrow{f} \underline{h^0(B)} \\
\\
h^{-1}(A) \xrightarrow{f} h^{-1}(B)
\end{array}$$

h_{horiz} inherits d_v!! $\qquad h_{\text{vert}}$ inherits d_h

↖ easy exercises ↗

~~Exercise $h^k(\mathrm{Tot}^{\oplus}) = h^k(\mathrm{Cone}\, f)$ ≠ $\mathrm{Coker}(h^k(A) \xrightarrow{f} h^k(B)) \oplus \mathrm{Ker}(h^{k+1}(A) \xrightarrow{f} h^{k+1}(B))$ ... $h^k(\mathrm{Coker} f^\bullet) \oplus h^{k}(\mathrm{Ker} f^\bullet)$ ...~~

There is a relationship b/w $h_{vert}(h_{horz}(\ ))$, $h_{horz}(h_{vert}(\ ))$ and $h_{tot}(\ )$ but it is complicated: "a spectral sequence"

~~Ex~~ Consider $h_{horz}(h_{vert}(\ )) =$

$$\begin{matrix} 0 & \operatorname{Ker} f^1_* & \operatorname{Coker} f^1_* & 0 \\ 0 & \operatorname{Ker} f^0_* & \operatorname{Coker} f^0_* & 0 \end{matrix}$$

$f^i_* : h^i(A) \to h^i(B)$

← total degree 0

Claim: $h^k(\operatorname{Cone} f)$ admits a s.e.s.

$$0 \to \operatorname{Coker} f^k_* \to h^k(\operatorname{Cone} f) \to \operatorname{Ker}(f^{k+1}_*) \to 0$$

Consider $h^0(\operatorname{Cone} f)$. $\operatorname{Ker} d^0$ is $\begin{bmatrix} b \\ a \end{bmatrix} \in \begin{bmatrix} B^0 \\ A^1 \end{bmatrix}$ s.t. $d_B b = -fa$, $-d_A a = 0$

Then $a \in \operatorname{Ker} d^1_A$ so $\bar a \in h^1(A)$. $f^1_*(\bar a) = \overline{fa} \in h^1(B)$, but $fa = d_B b$ so $\overline{fa} = 0$.

$\Rightarrow \bar a \in \operatorname{Ker}(f^1_*)$.

If $\bar a \in h^1(A)$ is zero, $\exists a' \in A^0$, $-d_A a' = a \rightsquigarrow \begin{bmatrix} b \\ a \end{bmatrix}$ and $\begin{bmatrix} b' \\ 0 \end{bmatrix}$ agree in $h^0(\operatorname{Cone} f)$

and $\bar b'$ is well defined in $\operatorname{Ker} d_B / (\operatorname{Im} d_B + \operatorname{Im} f|_{\operatorname{Ker} d_A})$ ... exercise.

$h_{vert}(h_{horz}(\ )) =$

$$\begin{matrix} 0 & h^1(\operatorname{Ker} f) & h^1(\operatorname{Coker} f) & 0 \\ 0 & h^0(\operatorname{Ker} f) & h^0(\operatorname{Coker} f) & 0 \end{matrix}$$

Claim: There is a l.e.s.

$$\cdots \to h^{-1}(\operatorname{Cone} f) \to h^{-1}(\operatorname{Coker} f) \to h^1(\operatorname{Ker} f) \to h^0(\operatorname{Cone} f) \to h^0(\operatorname{Coker} f) \to h^2(\operatorname{Ker} f) \to h^1(\operatorname{Cone} f) \to \cdots$$

↑ whoa! Snake-lemma like thing... map of degree $(-1, +2)$

How to understand this relationship?

(Ex) B_i complexes, $\cdots \to B_0 \xrightarrow{f_0} B_1 \xrightarrow{f_1} B_2 \to \cdots$ a complex of complexes // ← chain maps

glue into a bicomplex

$$\begin{matrix} \uparrow & & \uparrow & & \uparrow \\ B^1_0 & \to & B^1_1 & \to & B^1_2 \\ d^0_{B_0}\uparrow & & -d_{B_1}\uparrow & & d_{B_2}\uparrow & \quad \uparrow -d_{B_3} \\ B^0_0 & \to & B^0_1 & \to & B^0_2 \\ \uparrow & & \uparrow & & \uparrow \end{matrix}$$

← w/o signs, no double complex. put $(-1)^i d_{B_i}$ as direct.

Relationship of cohomologies even worse.

Subex: Spse $0 \to A^\circ \to B^\circ \to C^\circ \to 0$ s.e.s. of complexes

$$\begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & A^1 & \to & B^1 & \to & C^1 & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & A^0 & \to & B^0 & \to & C^0 & \to & 0 \end{array}$$

$h_{horiz} = 0$! We will show $\Rightarrow h_{tot} = 0$.

h_{vert}:

$$0 \to h^1(A) \to h^1(B) \to h^1(C) \to 0$$

$$0 \to h^0(A) \to h^0(B) \to h^0(C) \to 0$$

$h_{horiz}(h_{vert}) =$

$$\begin{array}{ccc} Q^1 & 0 & Q^2 \\ Q^0 & 0 & Q^1 \\ Q^{-1} & 0 & Q^0 \end{array}$$

($\wr\wr$)

b/c of les $h^0(B) \to h^0(C) \to h^1(A) \to h^1(B)$

again, some kind of new map of degree $(-2,+1)$...

(3) $Tor^*(M,N)$ M right R-mod N left R-mod.

$P^\circ \to M$ right proj resn $Q^\circ \to N$ left proj resn.

$P^\circ \otimes N \to M \otimes N$ $M \otimes Q^\circ \to M \otimes N$

$$\begin{array}{ccccccc} P^2\otimes N & \to & P^1\otimes N & \to & P^0\otimes N & \to & M\otimes N \\ \uparrow & & \ominus\uparrow & & \uparrow & & \ominus\uparrow \\ P^2\otimes Q^0 & \to & P^1\otimes Q^0 & \to & P^0\otimes Q^0 & \to & M\otimes Q^0 \\ & & \ominus\uparrow & & \uparrow & & \ominus\uparrow \\ & & P^1\otimes Q^1 & \to & P^0\otimes Q^1 & & M\otimes Q^1 \\ & & & & \uparrow & & \ominus\uparrow \\ & & & & & & M\otimes Q^2 \end{array}$$

need signs!

$P^\circ \otimes Q^\circ$ tensor product double complex

Consider augmented bicomplex.

All rows exact except row 0 — they compute $Tor^*(M, Q^j) = 0$ since Q^j project

All cols exact except col 0, $Tor^*(P^i, N)$

$$h_{horiz} = \begin{array}{ccc|c} \cdots & Tor^2_{(M)} & Tor^1_{(M)} & 0 \\ \hline & & 0 & 0 \end{array}$$

$$h_{vert} = \begin{array}{c|c} 0 & 0 \\ \hline 0 & Tor^1_{(N)} \\ & Tor^2_{(N)} \end{array}$$

Claim: $h_{horiz} \cong h_{vert} \cong h_{tot}$ in each tot degree.

$Tor^i_{(M)}(M,N) \cong Tor^i_{(N)}(M,N)$

Pf: $x \in P^2 \otimes N$,

$$\begin{array}{ccccc} & & x & \mapsto & 0 \\ & & \uparrow & & \uparrow \\ \exists\, y & \mapsto & z & \mapsto & 0 \\ \uparrow & & \uparrow & & \\ \exists\, w & \mapsto & v & \mapsto & 0 \\ \uparrow & & \uparrow & & \\ \exists\, u & \mapsto & t \in M\otimes Q^2 & & \end{array}$$

gives well-defined map $Tor^2 \to Tor^2$ indep of choices!!

Zig-zag / staircases.

Ready for the general theory yet?

Take a bicomplex $A^{\bullet\bullet}$. From this we get:

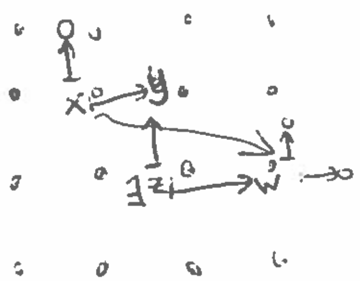
⓪ $E_0^{\bullet\bullet}$ this is just $A^{\bullet\bullet}$ w/ only d_{vert} remembered $\curvearrowright d_0$ degree $(0,+1)$

⇓ take cohomology

① $E_1^{\bullet\bullet}$ is $h_{vert}(A^{\bullet\bullet})$, has differential induced from d_{horiz} $\curvearrowright d_1$, degree $(+1,0)$

⇓ take cohomology

② $E_2^{\bullet\bullet}$ is $h_{horiz}(h_{vert}(A^{\bullet\bullet}))$, has induced differential d_2 of degree $(+2,-1)$



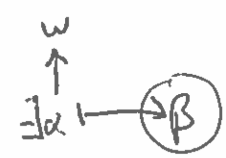
$x \mapsto w$ descends to well defined map, independent of choices.

$h_{horiz} h_{vert} A^{pq} \mapsto h_{horiz} h_{vert} A^{p+2,q-1}$

⇓ take cohomology

③ $E_3^{\bullet\bullet}$ is $h_{(2,-1)} h_{(1,0)} h_{(0,1)}(A^{\bullet\bullet})$, has induced differential of degree $(+3,-2)$

i.e. as above but $x \mapsto 0$ means w is in image



take β instead.

⇓ etcetera.

The collection of all these pages and differentials is the spectral sequence attached to the bicomplex $A^{\bullet\bullet}$, having chosen vert then horiz.

If you do horiz then vert, get differentials of degrees $(+1,0)$ $(0,+1)$ $(-1,+2)$ etc...

If $A^{\bullet\bullet}$ has certain boundedness conditions, $d_k = 0$ both to and from any given A^{pq} for fixed p,q, $k \gg 0$

This implies $E_K^{pq} = E_{K+1}^{pq} = E_{K+2}^{pq}$ $\forall k > K$, call this E_∞^{pq}.

What is the relationship b/w $E_\infty^{\bullet\bullet}$ and $h_{tot}(A^{\bullet\bullet})$?

(Assuming boundedness conditions)
We say $E_\bullet^{pq}$ <u>converges</u> to $h^\bullet_*$ if h^k has a filtration w/ subquotients E_∞^{ij}, $i+j=k$, in order of i, that is h^k has filtration

$$0 \subset \dots \subset F_{i+1}h^k \subset F_i h^k \subset \dots = h^k$$

s.t. $F_i h^k / F_{i+1} h^k \cong E_\infty^{ij}$, $j=k-i$. (This filtration is finite w/ boundedness conditions)

(If not finite, also have to ask if $\bigcap F_i h^k = 0$ and $\bigcup F_i h^k = h^k$ etc, so $\exists$ weaker notion of convergence when these fail. We'll ignore this shit)

<u>Thm</u> (Assuming boundedness conditions) $E_\bullet^{pq}$ of a double complex converges to [redacted] $h^\bullet(\mathrm{Tot}^\oplus A^{\bullet\bullet})$.

Ex: We said

$$0 \to \mathrm{Coker}\, f_X^k \to h^k(\mathrm{Cone}(f)) \to \mathrm{Ker}(f_X^{k+1}) \to 0$$

$\mathrm{Coker}\, f_X^k = E_2^{0k} = E_\infty^{0k}$, $\mathrm{Ker}(f_X^{k+1}) = E_2^{-1\,k+1} = E_\infty^{-1\,k+1}$

$F_{-1} = F_{-15} = h^k$

$F_0 = \mathrm{Coker}\, f_X^k$

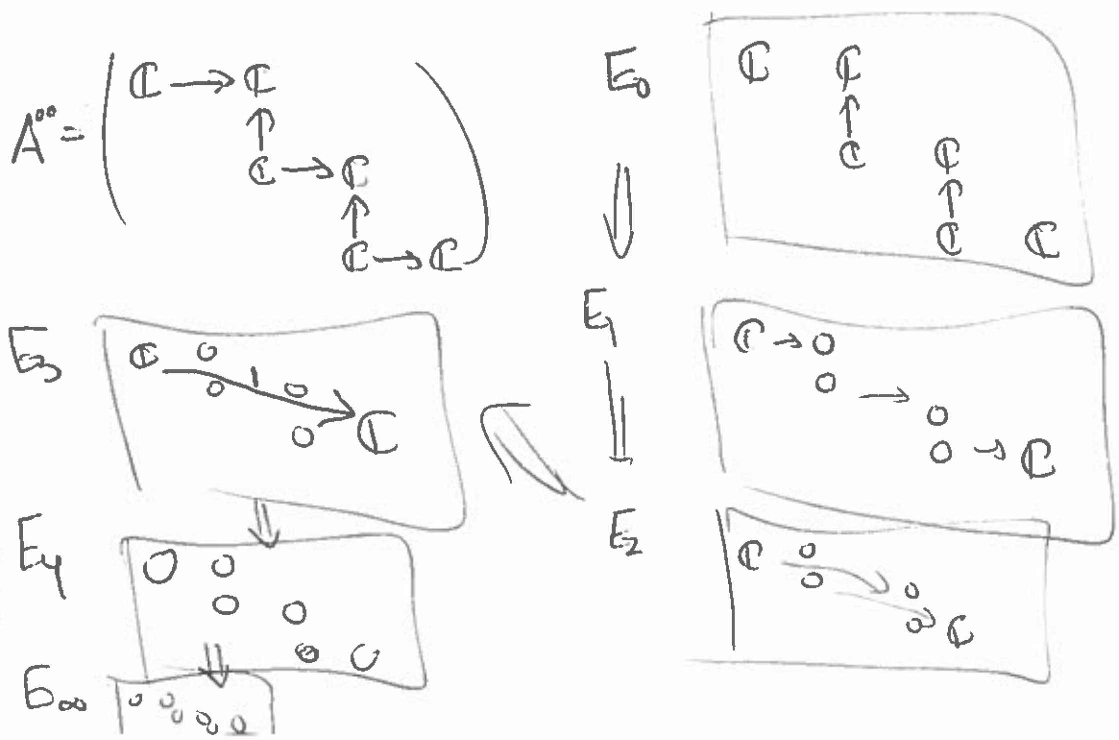
$F_{15} = F_1 = 0$

<u>Superimportant</u>: Can NOT use this to compute $h^\bullet(\mathrm{Tot}^\oplus A)$ naively — only get the subquots in a filtration, not the object itself.

If we're in a semisimple category like Vect, it is good enough though ($\mathbb{Z}$-mod is much harder!)

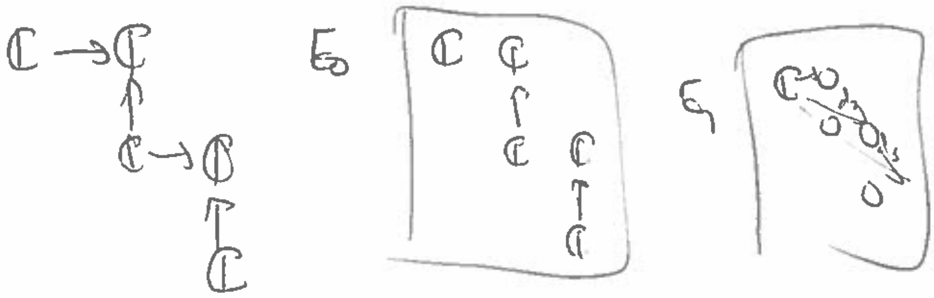
[redacted] If Exts vanish, same result.

<u>Examples</u>: ①



d_3 not obvious from E_2, need to rewind all the way to $A^{\bullet\bullet}$!!!

(2)

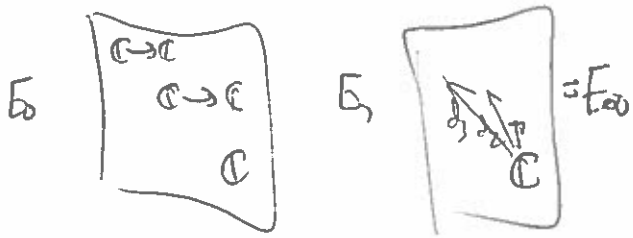


$= E_\infty$

$\Rightarrow h^0_{tot} = \mathbb{C}$

$h^i_{tot} = 0 \quad i \neq 0$

do horiz then vert

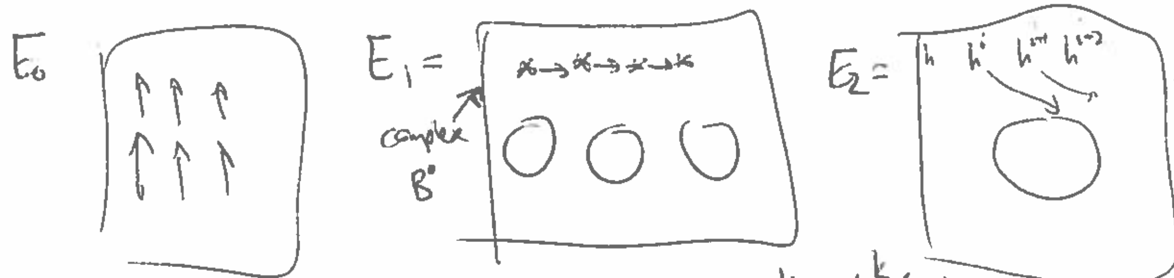


$= E_\infty$

$\Rightarrow h^0_{tot} = \mathbb{C}$

$h^i_{tot} = 0$

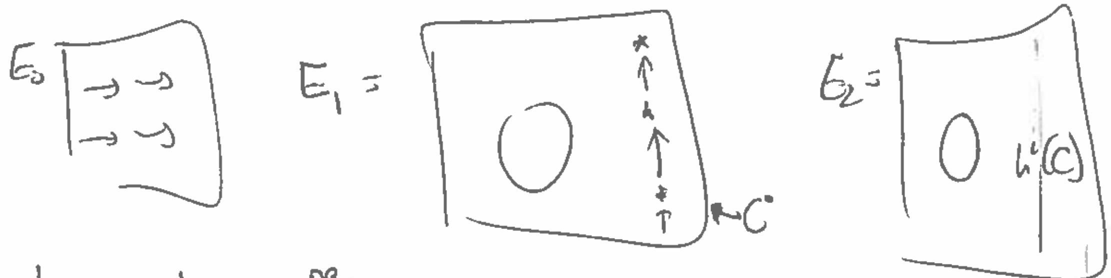
(3) Some complex where h_{horiz} concentrated in one column, h_{vert} in one row



$\Rightarrow E_\infty$

Filtration on h_{tot} has only one term so $h^k_{tot} = h^k(B^\cdot)$

For horiz



$= E_\infty$ $\quad h^k_{tot} = h^k(C^\cdot)$

so $h^k(C^\cdot) \cong h^k(B^\cdot)$

Tor/Ext balancing

Do w/ ... complexes ...

(4) If at any point E^{pq}_k <u>vanishes like a checkerboard</u> then $E^{pq}_k = E^{pq}_\infty$

b/c all differentials have odd total degree

Common in complex geometry for this to happen.

<u>Parity miracle</u>!

<u>Utility/Examples</u> (1) <u>Leray-Serre spectral sequence</u> $F \to E \to B$ fiber bundle, $\pi_1(B) = 0$ (or more generally, Serre fibration)

$\exists$ spectral sequence w/ $E^{pq}_2 = H^p(B; H^q(F)) \Longrightarrow H^{p+q}(E)$

Where's E_0? E_1? Can find the simplicial chain complexes, but more natural to take cohomology & skip a few steps. Makes computing d_2, d_3, etc difficult though!!

What is $H_p(B; H_q(F))$? $B \ni x \mapsto H_q(\pi^{-1}(x))$ is a vector bundle, local coeff system. (trivial since $\pi_1(B)=0$)

~~If we look over $\mathbb{C}$, (no higher Tor) they're just $H_p(B)\otimes H_q(F)$~~

Univ Coeff Thm: $H_p(B; H_q(F)) \cong H_p(B)\otimes H_q(F) \oplus \mathrm{Tor}_1(H_{p-1}(B), H_q(F))$

If coeffs are over $\mathbb{C}$, all Tor_1 vanish.

Ex: $\mathbb{C}^* \supset S^1 \to S^3 = SU(2)$, $\downarrow S^2$

$E_2 =$ (rows $H^*(S^1)$, columns $H^*(S^2)$):

$\mathbb{Z}\ 0\ \mathbb{Z}$
$\mathbb{Z}\ 0\ d_2\ \mathbb{Z}$

$E_3 =$
$0\ \ \mathbb{Z}$
$\mathbb{Z}\ \ 0$
$= E_\infty$

$H^*(S^3) = \begin{cases} \mathbb{Z} & i=0,3 \\ 0 & \text{else.} \end{cases}$

Ex: $\mathbb{P}^1_{\mathbb{C}} \to Fl(0,1,2,3)_{\mathbb{C}}$, $\downarrow Gr(1,3) = \mathbb{P}^2_{\mathbb{C}}$

$E_2 =$
$\mathbb{C}\ 0\ \mathbb{C}\ 0\ \mathbb{C}$
$0\ 0\ 0\ 0\ 0$
$\mathbb{C}\ 0\ \mathbb{C}\ 0\ \mathbb{C}$
$= E_\infty$

Vanishes like a checkerboard! $H^*(Fl(0,1,2,3)) \cong H^*(\mathbb{P}^1)\otimes H^*(\mathbb{P}^2)$

(Can compute ring structure w/ spectral sequence too...)

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2) Groth Spectral Seq $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ right exact $\Rightarrow$ GF right exact.

Compare $L^jG \circ L^iF$ with $L^{i+j}(G\circ F)$

Thm (Groth): Suppose F sends projectives to G-acyclics. Then $\exists$ spectral sequence

w/ $E_2^{ij} = L^iF(L^jG(M)) \Rightarrow L^{i+j}(G\circ F)(M)$

Pf: Choose $P^0 \twoheadrightarrow M$, and apply F. Now choose proj res'ns in $\mathcal{B}$ of each, which give into a big complex. Why can we do this? Soon.

$\to F P^{-1} \to F P^0 \to F(M)$
$X^{-1,0} \to X^{00} \to P^0$
$\to X^{0,-1} \to P^{-1}$
P^{-2}

Then apply G.

$GX^{-1,0} \to GX^{00} \to GP^0 \to GF(M)$
$\to GX^{0-1}$

② Hyper-derived functors ... a terrible name

How did derived functors work?

- took $M \in \mathcal{A}$
- viewed as a complex $0 \to M \to 0$
- found a quasiiso $\begin{array}{ccccccc} P^{-2} & \to & P^{-1} & \to & P^0 & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & \to & M & \to & 0 \end{array}$
- applied functor to P° instead
- took homology

What if you replaced with: Start with a bdd complex $(M^\circ)_\circ$

$$\begin{array}{ccccccccc} (0 \to M^{-d} \to \dots \to & M^{-2} & \to & M^{-1} & \to & M^0 & \to & 0) \\ & \uparrow & & \uparrow & & \uparrow & & \\ \dots & P^{-2} & \to & P^{-1} & \to & P^0 & \to & 0 \end{array}$$

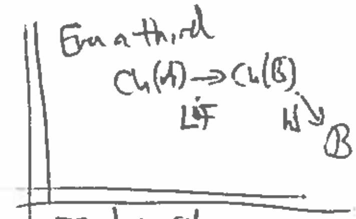
- find a qiso
- continue. Exercise: Conditions if $\exists$ proj.

This is called the hyper derived functor $\mathbb{L}^i F(M^\circ)$. I'm happy if you just call it a derived function

$$\mathbb{L}^i F : Ch(\mathcal{A}) \to \mathcal{B}$$

But there's another thing you can do: $M^\circ \in Ch(\mathcal{A}) \longrightarrow \mathcal{B}$, h^i, $h^i(M^\circ) \in \mathcal{A}$, $L^j F(h^i(M))$

Even a third: $Ch(\mathcal{A}) \to Ch(\mathcal{B})$, $L^j F$, h^i, $\mathcal{B}$



First tools set up.

Relationship: this is exactly the kind of situation where you got a spectral sequence

Def: $\mathcal{A}$ has enough proj. $M^\circ \in Ch(\mathcal{A})$. A Cartan–Eilenberg res'n of M° is an (augmented) double complex $P^{\circ\circ}$ satisfying:

$$\begin{array}{ccccccc} \dots & M^{-2} & \to & M^{-1} & \to & M^0 & \to \\ & \uparrow & & \uparrow & & \uparrow & \\ & P^{-2,0} & \to & P^{-1,0} & \to & P^{0,0} & \\ & & & \uparrow & & \uparrow & \\ & & & P^{-1,-1} & \to & P^{0,-1} & \end{array}$$

- ⊖ P^{ij} all projective
- ⓪ if $M^i = 0$ then the whole column is zero.
- ① The following are projective resolution, for each column!
  - a) $P^{i*} \to M^i$
  - b) $\operatorname{Ker} d_h \subset P^{i*} \to \operatorname{Ker} d_h \subset M^i$
  - c) $\operatorname{Im} d_h \subset P^{i*} \to \operatorname{Im} d_h \subset M^i$
  - d) $h^{i*}_{hor} \to h^i(M)$

In particular, all these things are summands, $P^{\circ\circ}$ is very split.

(remember the horseshoe lemma? Very similar.)

Rmk: Only need to assume b), d) and rest follow.

Thm: CE res'ns exist. **Pf:** Horseshoe over and over.

Moreover, any maps of complexes $M^\cdot \to N^\cdot$ induce maps of CE res'ns, $P^{\circ\circ} \to Q^{\circ\circ}$, unique up to homotopy of bicomplexes (see book)

Moreover, $\mathrm{Tot}^{\oplus} P^{\circ\circ} \to M^\circ$ is a qiso, and homotopy of bicomplex ↦ homotopy of $\mathrm{Tot}^{\oplus}$;

Def: $\mathbb{L}^k F(M^\circ) = h^k\, \mathrm{Tot}_\oplus (FP^{\circ\circ})$ indep of choice of CE resln.

Thm: It's a functor, universal in some sense. Agrees with $L^iF(M)$ for complex $0 \to M \to 0$

Hard Ded Given SES $0 \to A^\circ \to B^\circ \to C^\circ \to 0$ get les $\ldots \mathbb{L}^i F(A^\circ) \to \ldots$.

Thm: There is a convergent spectral sequence w/

$$(*)\quad E_2^{pq} = L^pF\left(h^q(M^\circ)\right) \Longrightarrow \mathbb{L}^{p+q}F(M^\circ)$$

If M bdd above, can also do the opposite

$$(**)\quad E_2^{pq} = h^p\left(L^qF(M^\circ)\right) \Longrightarrow \mathbb{L}^{p+q}F(M^\circ).$$

Having two is great. Some are friendly, some are not. One lets you compute E_∞. Other is interesting.

Pf: Consider $F(P^{\circ\circ})$.

$$\left(\begin{array}{ccccc} FM^{-2} & \to & FM^{-1} & \to & FM^0 \ldots \\ \uparrow & & \uparrow & & \uparrow \\ FP^{-2\,0} & \to & FP^{-1\,0} & \to & FP^{00} \to \\ & & \uparrow & & \uparrow \\ & & FP^{-1\,-1} & \to & FP^{0\,-1} \to \end{array}\right)$$

By defn, $h^k_{tot}(FP^{\circ\circ}) = \mathbb{L}^kF(M^\circ)$

Take hvert, get $L^jF(M^i) \longrightarrow L^jF(M^{i+1}) \to \ldots$

Take hhoriz get $h^p\left(L^qF(M^\circ)\right)$. This is for a bicomplex like

If M bdd above, it looks like

and is regular.

Take hhoriz, before apply F get

$$\begin{array}{c} h^j(M^\circ) \\ \uparrow \\ h^j(P^{\circ 0}) \\ \uparrow \\ h^j(P^{\circ -1}) \\ \uparrow \\ \vdots \end{array}$$

but since these were all summands, they are still all summands

After apply F, and get

$$\begin{array}{c} h^j(FP^{\circ 0}) \\ \uparrow \\ h^j(FP^{\circ -1}) \\ \uparrow \\ \vdots \end{array}$$

which resolves $F(h^j(M))$, not $h^j(FM)$

So now hvert gives $L^pF(h^q(M))$.

This is for bicomplex

which is not regular but is good enough for our purposes.

③ **Groth spectral seq** $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ both right exact. Assume $\exists$ proj.

Thm (G) Suppose F sends $\mathrm{Proj}\,\mathcal{A}$ to G-acyclics. Then $\exists$ spec. seq. with

$$E_2^{pq} = L^pG(L^qF(M)) \Longrightarrow L^{p+q}(G\circ F)(M)$$

(Leray–Serre is example of this.) $Sh_c(E) \xrightarrow{\pi_*} Sh_c(B) \xrightarrow{\Gamma} Sh_c(pt)$, with the composite Γ running from $Sh_c(E)$ to $Sh_c(pt)$.

$H^*(E) = R\Gamma(\underline{\mathbb{C}}_E)$
$H^*(B) = R\Gamma(\underline{\mathbb{C}}_B)$
$H^*(B; H^*(F)) = R\Gamma(R\pi_*(\underline{\mathbb{C}}_E))$

Pf: $P^\bullet \twoheadrightarrow M$, apply F, get $(FP^\bullet) \to FM$.

Now take $\mathbb{L}G(FP^\bullet)$ via
$$\begin{pmatrix} FP^\bullet \to FM \\ \uparrow \\ Q^{\bullet\bullet} \end{pmatrix} \xrightarrow{G} \begin{matrix} GFP^\bullet \to GFM \\ \uparrow \\ GQ^{\bullet\bullet} \end{matrix}$$

S.S. (✱✱) gives $E_2^{pq} = h^p(L^qG(FP^\bullet)) \Longrightarrow \mathbb{L}^{p+q}G(FP^\bullet)$

Now $L^qG(FP^\bullet) = 0$ for $q>0$, since each FP^i is G-acyclic. Concentrated in $q=0$, where it is just $h^p(GFP^\bullet) \equiv L^p(GF)(M)$. So converges immediately at E_2, and $\mathbb{L}^{p+q}G(FP^\bullet) = L^{p+q}(GF)(M)$.

Now (✱) gives $E_2^{pq} = L^pG(h^q(FP^\bullet)) = L^pG(L^qF(M))$, as desired. ☑.

Many, many, many examples of this!
(✱✱) is friendly. (✱) is not.

Khovanov-style Consider bicomplexes in $\mathrm{Vect}_{\mathbb{k}}$. These are bigraded modules for $\Lambda^*(\mathbb{k}^2)$, a Frob alg $\Rightarrow$ proj = inj.

Free: $\begin{matrix} \mathbb{k} \xrightarrow{1} \mathbb{k} \\ 1\uparrow \quad \uparrow -1 \\ \mathbb{k} \xrightarrow{1} \mathbb{k} \end{matrix}$, so if ever $v \mapsto w \neq 0$, $\mapsto -w$, then get copy of free, as a summand.

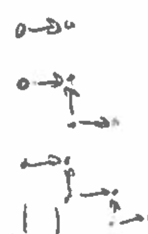
Indecomposables: projectives and staircases (all $\mathbb{k}$, all maps 1).

Need only see how S.S. behaves for each staircase.

Odd length: $E_1 = \begin{matrix} 0 \\ 0\ 0 \\ 0\ \mathbb{k} \end{matrix} = E_\infty$ (no higher differentials)

Even 1: $E_1 = \begin{matrix} 0 \\ 0\ 0 \\ 0 \end{matrix}$

Even 2: E_1 $\begin{matrix} \mathbb{k}\ \mathbb{k} \\ \mathbb{k}\ 0 \\ 0\ \mathbb{k} \end{matrix}$ $\quad \begin{matrix} \mathbb{k}\ 0 \\ 0\ 0 \\ 0\ \mathbb{k} \end{matrix}$



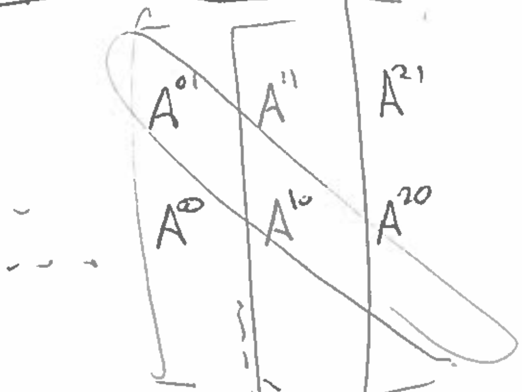
Interesting!

Each E_2-II indecomp contributes to precisely one differential. The passage from $E_2 \rightsquigarrow E_\infty$ just kills copies of one indecomposable at a time, essentially. 52
(No worries about filtration here — for Vect$_k$ everything splits).

Filtered complexes + convolution

Filtered complexes + convolutions

Given a bicomplex $A^{\bullet\bullet}$ have sub-bicomplexes



$$\cdots \supset A^{\geq 0\,\bullet} \supset A^{\geq 1\,\bullet} \supset A^{\geq 2\,\bullet} \supset \cdots$$

This induces sub-complexes

$$\cdots \supset \mathrm{Tot}^{\oplus} A^{\geq 0} \supset \mathrm{Tot}^{\oplus} A^{\geq 1} \supset \mathrm{Tot}^{\oplus} A^{\geq 2} \supset$$

($F_0 \mathrm{Tot}^{\oplus}$, $F_1 \mathrm{Tot}^{\oplus}$)

Even though $A^{\bullet\bullet}$ is added one column at a time, every homological degree of $\mathrm{Tot}^{\oplus}$ is added to at once.

← the filt on $(\mathrm{Tot}^{\oplus} A^{\bullet\bullet})^{k}$

What we get is a filtration on the complex $\mathrm{Tot}^{\oplus} A^{\bullet\bullet}$!

Indexing annoyance: $\left((F_i \mathrm{Tot}^{\oplus})^{\bullet} / (F_{i+1} \mathrm{Tot}^{\oplus})^{\bullet} \right)^{k} \cong A^{i\,(k-i)}$

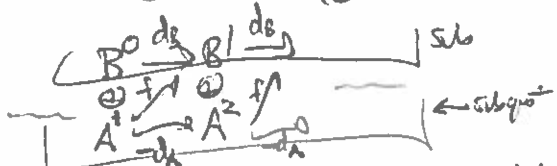
This subquotient complex is the column $A^{i\bullet}$ of A... but shifted! The 0^{th} diagonal becomes the 0^{th} homol degree.

$\cdots \supset F_i C^{\bullet} \supset F_{i+1} C^{\bullet} \supset \cdots$

What kind of filtered complexes arise in this way? Only very special ones!

Condition 1: It is (degreewise split filtered), i.e. the filtration on hom. degree k is split $\forall k$ (each F_i is $F_{i+1} \oplus$ something) (but not a direct sum of complexes!!)

Ex: $\mathrm{Cone}(f) =$



is degreewise splitfiltered but not split filtered.

A degreewise splitfiltered filtered complex is called a convolution.

The general idea of a degreewise splitfiltered filtered complex is ... indexed by a poset I. (Usually $\mathbb{Z}$)

Def: Let $(P_i, d_{P_i})_{i \in I}$ be a collection of complexes indexed by a poset I.

A convolution of this data is a complex $(C^{\bullet}, d)$ where

$$C^{k} = \bigoplus P_i^{k} \qquad \text{and so} \qquad \underset{C^k \to C^{k+1}}{d^{k}} = \sum \underset{P_i^k \to P_j^{k+1}}{d_{ij}^{k}} \qquad \text{where}$$

- $d_{ii} = d_{P_i}$
- $d_{ij} = 0$ for $j < i$

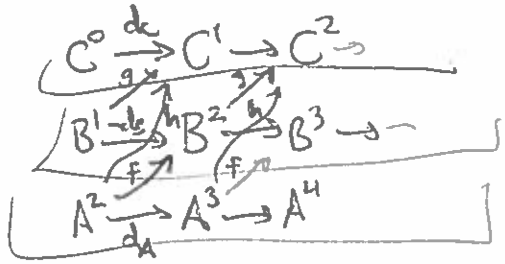
Then $(F_i C^{\bullet})^{k} = \bigoplus_{j \geq i} P_j^{k}$ is a subcomplex, giving a degreewise splitfiltration w/ subquots (P_i, d_{P_i})

Ex: $\mathrm{Cone}(f)$ is a convolution of $P_0 = (B, d_B) = B$ and $P_{-1} = A[1]$ ← (w/ $d_{A[1]} = -d_A$.)

$$d_{00} = d_B \quad d_{-1-1} = d_{A[1]} = -d_A \quad d_{-10} = f \quad d_{0-1} = 0$$

Rmk: All convolutions w/ $I = \{-1, 0\}$ are cones! $\mathrm{Cone}(d_{-10})$ and condition that $d_C^2 = 0$ implies d_{-10} is a chain map $A^{\bullet} \to B^{\bullet}$!!

What about a convolution w/ 3 pieces? $f = d_{-2\,-1}$ $g = d_{1\,0}$ $h = d_{-2\,0}$



$d^2 = 0 \implies$ • $d_A^2 = d_B^2 = d_C^2 = 0$

• f is a chain map

• g is a chain map

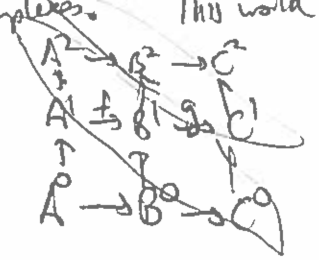
• $hd_A + hd_A + gf = 0$, i.e.

$-gf \simeq 0$ and h is a nullhomotopy of it !!!

So a convolution comes from $A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C^\cdot$ a <u>homotopy complex</u>, $d^2 \neq 0$ but $d^2 \simeq 0$.

but w/ fixed choice of nullhomotopy!!

Convolutions w/ 3 pieces where $d_{-2\,0} = 0$, i.e. $h = 0$, i.e. $gf = 0$ come from $A^\cdot \to B^\cdot \to C^\cdot$ actual complex of complexes. This would be $\mathrm{Tot}^{\oplus}$ of



But with $h \neq 0$, don't get $\mathrm{Tot}^{\oplus}$ of any bicomplex!!

<u>Condition 2:</u>

A convolution is <u>purely degree 1</u> if $I = \mathbb{Z}$ and $d_{ij} = 0$ unless $j = i+1$ or i

Easy dictionary: {purely degree 1 convolutions} $\longleftrightarrow$ {$\mathrm{Tot}^{\oplus}$ of bicomplexes}

<u>More rehashing:</u> If $A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C^\cdot$ w/ $gf = 0$ then get induced maps $A^\cdot[1] \to \mathrm{Cone}(g)$
$\mathrm{Cone}(f) \to C^\cdot$

Taking cone of them, get the full convolution (Either one, they're naturally isomorphic)

The map $A^\cdot[1] \to \mathrm{Cone}(g)$ only hits the $B^\cdot$ layer / the map $\mathrm{Cone}(f) \to C^\cdot$ is zero on the $A^\cdot$ layer

which is equivalent to saying $gf = 0$

But there are maps $A^\cdot \xrightarrow{f} B^\cdot$ and maps $\mathrm{Cone}(f) \to C^\cdot$ which are nonzero on both $A^\cdot$ and $B^\cdot$, and these give non-degree-1 convolutions when cone is taken.

Restricting to $B^\cdot \to C^\cdot$ get g with $-gf \simeq 0$ via homotopy h.

Anyway, there is also: the spectral sequence attached to a filtered complex!

$$E_0^{i\,(k-i)} = {}^k\!\left(F_iC/F_{i+1}C\right)$$ w/ induced differential d_0 from F_iC

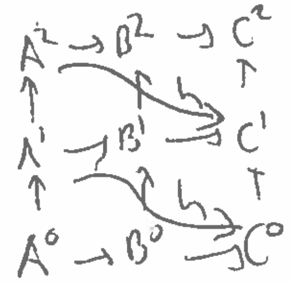
$$d_1 : h^k\left(F_iC/F_{i+1}C\right) \longrightarrow h^{k+1}\left(F_{i+1}C/F_{i+2}C\right)$$

is the boundary map in the l.e.s. for

$$0 \to F_{i+1}/F_{i+2} \hookrightarrow F_i/F_{i+2} \to F_i/F_{i+1} \to 0$$

d_2: worse...

But the point is that a convolution is "like" a bicomplex w/ higher differentials already!



d_2 is defined like before but also has an extra component coming from h!

(because not really a bicomplex, $d_{\text{horiz}}^2 \neq 0$, h is the necessary error term)

Details: swept blissfully away.

Thm: $(E_k^{p,q}, d_k) \Rightarrow h^*(C)$ under some boldness condition

Rmk: Not every filtered complex is degreewise split filtered, but on Vect or semisimple cat they are!

Many consequences, because applying a functor to a filtered complex/convolution yields a filtered complex/convolution!

Thm 1: G right exact, (F_iC) filtered complex w/ subquots (P_i), then $\exists$ spectral sequence w/

$$E_2^{pq} = \mathbb{L}^p G(P_q) \Rightarrow \mathbb{L}^{p+q}(C)$$

Thm 2: Observe that any complex $M^\bullet = (\to M^{-1} \to M^0 \to M^1 \to)$ has a filtration

with $F_iM = (0 \to 0 \to M^i \to M^{i+1} \to \_\_\_) = M^{\geq i}$

subquots are $0 \to M^i \to 0$ in degree i

Consequently $E_2^{pq} = \mathbb{L}^p G(M^q) \Rightarrow \mathbb{L}^{p+q}(M)$ (check indexing)

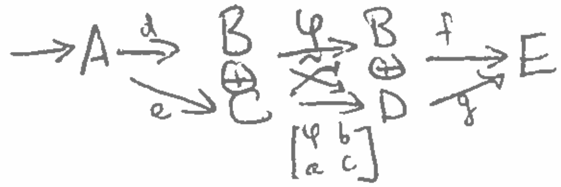
AGAIN: Key point: outside of algebraic topology (hard(!)), spectral sequences are mostly useless (computing higher differentials is hard) except when they degenerate at E_2 i.e. $E_2 = E_\infty$

Ex: In my recent paper w/ Hogancamp where I computed $HH^k(C^\bullet)$ ← special complex of bimodules by finding a description of $C^\bullet$ as a convolution of simpler terms, which each are convolutions of simpler, etc etc. HH^k(base case) is in even degrees only! $\Rightarrow E_2 = E_\infty$ so HH^k(next) is also in even degrees $\Rightarrow$ induction $\Rightarrow HH^k(C^\bullet) = \bigoplus HH^k(\text{pieces})$.

YAY. [illegible]

Robert's philosophy + Gaussian elimination

Suppose inside a complex we have a configuration like



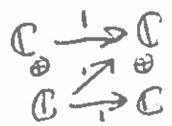
φ an isom. Then $B \xrightarrow{\varphi} B$ is neither a sub, quot, summand, [illegible] but nonetheless it should "contract" out, i.e. which is homotopy equivalent. But what are the maps?

$\exists$ complex $\to A \xrightarrow{e} C \xrightarrow{c - a\varphi^{-1}b} D \xrightarrow{g} E \to \dots$

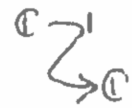
Exercise: They are!

This formula/process is called Gaussian elimination. Reducing the complex.

Ex:



contract out $\nearrow$ and get



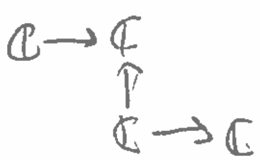
(even though originally there was no map b/w those summands)

This is a really effective tool for computation, I use it often.

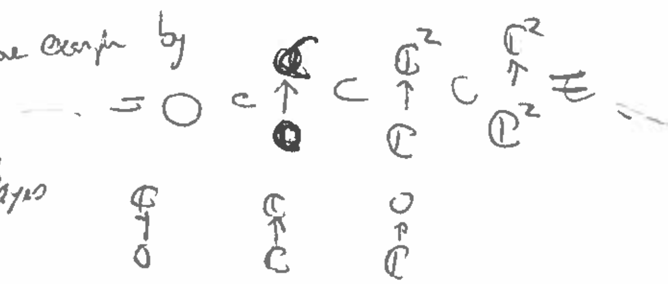
Note: In Ch(Vect) or any semisimple cat, every nonzero map is an isom, so can repeatedly Gaussian eliminate until all differentials are zero and only the cohomology remains. But this is not just chopping off isoms, since you can create new isoms, as in the example above.

Suppose you have a convolution / split filtered complex of $V_{i,j}$ (bigraded). When doing Gaussian elimination, the terms which cancel may be in different filtered degrees. One algorithm: Gaussian eliminate w/in each layer first (d_{ii} on P_i). First step is just taking h_{vert}, then h_{horiz}, then … Then $d_{i\,i+1}$ then $d_{i\,i+2}$ etc. ← not the original, but the new differential after GE.

Ex:



is filtration of above example by



w/ layers

Applying first GE is just taking h_{vert}.

get

, [illegible]

Idea/point: The differential on the E_k page is the stuff that is eliminated b/w i^{th} and $(i+k)^{th}$ layer. (This is iterative, depends on previous elimination — original complex above had $d_{02} = 0$)

When you've finished all GE, end up with $h^*(\text{complex}) = h^*(Tot^{\oplus}) = E_\infty$.