

Towards triangulated categories

Gaussian elimination etc.

(1)

Warning!

Def: A complex C^\bullet is contractible if id_C is nullhomotopic, i.e. if $C \simeq 0$.

Ex: $0 \rightarrow B \xrightarrow{id_B} B \rightarrow 0$ is contractible, w/ homotopy h .

Now if C^\bullet is contractible then $1 = dh + hd$. Claim: dh, hd are orthogonal idempotents!

Pf: $dhhd = d(1-dh)h = dh + d^2h^2 = dh$ $hdhd = 0$ etc.

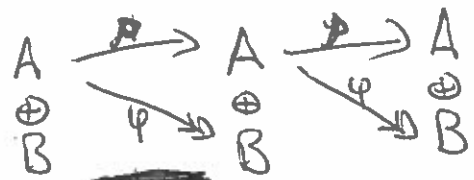
In example, dh projects to 0 and B
 hd projects to B and 0 .

Note: dh, hd are NOT chain maps

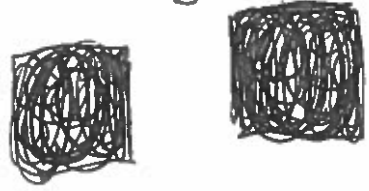
$d(dh) = 0$
 $(dh)d = (dh+hd)d = d$

but they do give direct summand decomps of each chain object. $C^i = A^i \oplus B^i$

$d=0$ on B .



But $d^2=0 \Rightarrow dp=0$
 $\Rightarrow hdp=0 \Rightarrow dhdp=0$
 $\Rightarrow p=0$



$\Rightarrow \varphi$ is isom, h is inverse isom.
 (a little more work)

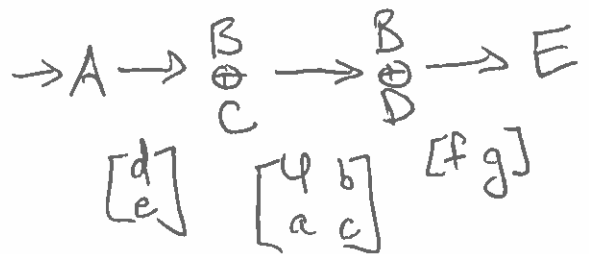
i.e. $C = \bigoplus 0 \rightarrow B \xrightarrow{id} B \rightarrow 0$ w/ various shifts.

Thm: In any additive cat, $0 \rightarrow B \xrightarrow{id} B \rightarrow 0$ are the only indecomp. contractible complexes.

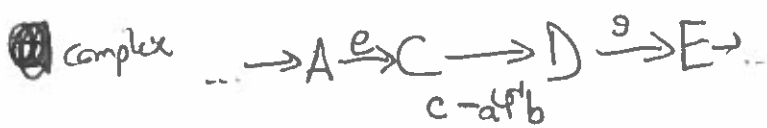
How do you "contract out" the contractible pieces?

Warning: Contractible \Leftrightarrow exact BUT exact $\not\Rightarrow$ contractible!
 $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$

Thm (Gaussian Elimination): Spoke inside a complex we have



where φ is an isomorphism. Then the



is homotopy equivalent to the original.

Rank: $B \xrightarrow{\sim} B$ is neither a sub, quotient, or summand, but still it contracts out! (2)

Ex: $C \xrightarrow{1} C \xrightarrow{\sim} B$
 $\oplus \nearrow \oplus$
 $B = C \xrightarrow{1} C$
 contract out B and get $C \xrightarrow{1} C$ even though originally there was no map b/w these summands.

This is a really effective tool for simplifying complexes.

Pf Sketch: Step ① Show that the original complex is isomorphic to a complex

$$A \longrightarrow \begin{matrix} B \\ \oplus \\ C \end{matrix} \longrightarrow \begin{matrix} B \\ \oplus \\ B \end{matrix} \longrightarrow E, \text{ i.e. } A \longrightarrow \begin{matrix} C \\ \oplus \\ B \end{matrix} \longrightarrow E$$

$$\begin{bmatrix} a \\ e \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & \oplus \end{bmatrix} \quad \begin{bmatrix} 0 & g \end{bmatrix}$$

$C = \text{coker } B$

this is "gaussian elimination", i.e. row reduction.

Step ② Contract out $B \xrightarrow{\sim} B$ w/ homotopy.

The scoop on cones $A \xrightarrow{f} B \rightsquigarrow \text{Cone}(f) = \begin{pmatrix} \rightarrow B^0 \xrightarrow{d_0} B^1 \rightarrow \\ \nearrow f \uparrow \oplus \nearrow f \\ \rightarrow A^1 \rightarrow A^2 \\ \downarrow -d \end{pmatrix}$

Facts: ① $0 \rightarrow B \xrightarrow{u} \text{Cone}(f) \xrightarrow{v} A[1] \rightarrow 0$ a seq. of complexes, w/ $vu=0$.

② $\text{Cone}(u) = \begin{pmatrix} B^1 \xrightarrow{d} B^2 \rightarrow \\ \searrow \text{id} \downarrow \nearrow \text{id} \\ B^0 \xrightarrow{d} B^1 \xrightarrow{d} B^2 \\ \nearrow f \uparrow \\ A^1 \xrightarrow{d} A^2 \rightarrow \\ \downarrow -d \end{pmatrix} \cong_{\text{G.E.}} A[1]$

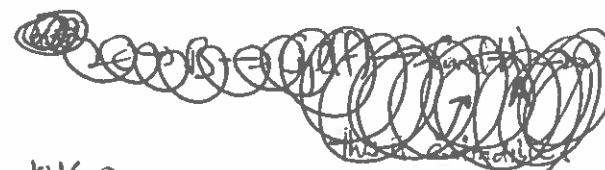
③ $\text{Cone}(v) = \begin{pmatrix} B^1 \xrightarrow{d} B^2 \rightarrow \\ \nearrow f \uparrow \\ A^2 \xrightarrow{d} A^3 \rightarrow \\ \searrow \text{id} \downarrow \nearrow \text{id} \\ A^1 \xrightarrow{-d} A^2 \rightarrow \end{pmatrix} \cong B[1].$ So $\text{Cone}(v) \cong B[1]$
 $\cong \text{Cyl}(f)$
 is homotopic to B .

④ Composita $A \xrightarrow{f} B \xrightarrow{u} \text{Cone}(f)$ is nonzero, but u nullhomotopic
 $\text{c.m.} \rightarrow \text{A} \xrightarrow{f} \text{B} \xrightarrow{u} \text{Cone}(f) \rightarrow \text{R} \xrightarrow{\text{A}[1]} \text{R}$

(5) Applying (1) to $\text{Cone}(V)$ get more s.e.s

$$0 \rightarrow A \rightarrow \text{Cyl}(F) \rightarrow \text{Cone}(F) \rightarrow 0$$

$$0 \rightarrow \text{Cone}(F) \xrightarrow{\text{is } A[1]} \text{Cone}(A) \rightarrow B[1] \rightarrow 0$$



(6) In l.e.s. of $\text{Cyl}(F)$

$$h^k(B) \rightarrow h^k(\text{Cone}(F)) \rightarrow h^k(A[1]) \xrightarrow{\delta} h^{k+1}(B)$$

$$h^k(A) \xrightarrow{f_*} h^k(B)$$

the triangle commutes. Was on exercise.

(7) In l.e.s. of $\text{Cone}(F)$

$$h^k(A) \rightarrow h^k(\text{Cyl}(F)) \xrightarrow{\text{is}} h^k(B)$$

is also f_* .

putting this together: the l.e.s. of $0 \rightarrow B \rightarrow \text{Cone}(F) \rightarrow A[1] \rightarrow 0$

$$\dots \rightarrow h^k(A) \rightarrow h^k(B) \rightarrow h^k(\text{Cone}(F)) \rightarrow h^{k+1}(A) \rightarrow \dots$$

is the same as the l.e.s. of $0 \rightarrow A \xrightarrow{f} \text{Cyl}(F) \rightarrow \text{Cone}(F) \rightarrow 0$

$$0 \rightarrow \text{Cone}(F) \xrightarrow{\text{is } A[1]} \text{Cone}(A) \rightarrow B[1] \rightarrow 0$$



We think of these s.e.s. as "rotated" of each other - cycle the three terms, but then replace them w/ hom. eq. things to get a s.e.s. of complexes.

(Can't just cycle naively: $A \xrightarrow{f} B \xrightarrow{u} \text{Cone}(F)$ Composition is not even zero, f not injective, u not surjective etc etc.)

This is the key idea, but first some major warnings about cones.

(8) Cones are NOT "canonical". What are the maps

$$\begin{array}{ccccccc} 0 & \rightarrow & B & \rightarrow & \text{Cone}(F) & \rightarrow & A \rightarrow 0 \\ & & \parallel & & \downarrow \varphi/\psi & & \parallel \\ 0 & \rightarrow & B & \rightarrow & \text{Cone}(F) & \rightarrow & A \rightarrow 0 \end{array} \quad ?$$

$$\varphi = \begin{pmatrix} \text{id}_B & h \\ 0 & \text{id}_A \end{pmatrix}$$

$$h: A^1 \rightarrow B^0$$

$$A^2 \rightarrow B^1$$

to be a chain map compute

$$dh + hd = 0, \quad -dh = hd$$

i.e. h is a chain map $A \rightarrow B[-1]$.

$$\Psi \simeq \text{Id}_{\text{Core}} \iff h \simeq 0 \text{ as maps } A \rightarrow B[-i],$$

so even in the homotopy category, a non-nullhomotopic map $A \rightarrow B[-i]$ (think: an element of $\text{Ext}^{-1}(A^0, B^0)$) gives a non-trivial automorphism of $\text{Core}(f)$.

- ⑨ It is obvious that
- a) f q-isom $\implies \text{Core}(f)$ exact from l.e.s.
 - b) f isom $\implies \text{Core}(f)$ contractible by S.E.

but if f is h.c.e., it need not be the case that $\text{Core}(f)$ is contractible!!

Exercise: A nullhomotopy for $\text{Id}_{\text{Core}(f)}$ is the same data as: $h = \begin{pmatrix} g & m \\ k & n \end{pmatrix}$

• a homotopy inverse k for f

• a homotopy g b/w fk and 1

• ~~h~~ in $1/w$ kf and 1

~~so~~

so $gf, -fn$ are both homotopies for fkf .

$\implies gf + fn$ is a homotopy for zero, i.e. a chain map $A \rightarrow B[-i]$

• ~~h~~ m b/w $gf + fn$ and 0 .
(need not exist!!)

Cones are great... but not perfect!! This causes some technical issues, but not too many.