Towards triangulated categories | Gaussian elimination etc.

Def: A complex $C^\bullet$ is contractible if id_C is nullhomotopic, i.e. if $C \simeq 0$. (homotopy equivalent)

Ex: $0 \to B \xrightarrow{\mathrm{id}_B} B \to 0$ is contractible, w/ homotopy h. $\mathrm{id}_B = h$

Now if $C^\bullet$ is contractible then $1 = dh + hd$. **Claim:** dh, hd are orthogonal idempotents!

Pf: $dhdh = d(1-dh)h = dh + \cancel{d^2}h^2 = dh$ $\qquad hddh = 0$ etc.

In example, dh projects to 0 and B
hd projects to B and 0.

Note: dh, hd are NOT chain maps
$d(dh) = 0$
$(dh)d \overset{*}{=} (dh+hd)d = d$.

but they do give direct summand decomps of each chain object. $C^i = \underset{\mathrm{Im}(hd)}{A^i} \oplus \underset{\mathrm{Im}(dh)}{B^i}$

$d = 0$ on B.

$$\begin{tikzcd} A \oplus B \arrow[r, "p"] \arrow[dr, "\varphi"'] & A \oplus B \arrow[r, "p"] \arrow[dr, "\varphi"] & A \oplus B \end{tikzcd}$$

But $d^2 = 0 \Rightarrow dp = 0$
$\Rightarrow hdp = 0 \Rightarrow \mathrm{id}_A p = 0$
$\Rightarrow p = 0$.

$\Longrightarrow$ φ is isom, h is inverse isom.
(little more work)

i.e. $C = \bigoplus\ 0 \to B \xrightarrow{\sim} B \to 0$ w/ various shifts.

Thm: In any additive cat, $0 \to B \xrightarrow{\sim} B \to 0$ are the only indecomp. contractible complexes.

Warning: contractible $\Rightarrow$ exact ★
BUT
exact $\not\Rightarrow$ contractible!
$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

How do you "contract out" the contractible pieces?

Thm (Gaussian Elimination): Sppse inside a complex we have

$$\to A \xrightarrow{\begin{bmatrix} d \\ e \end{bmatrix}} \begin{matrix} B \\ \oplus \\ C \end{matrix} \xrightarrow{\begin{bmatrix} \varphi & b \\ a & c \end{bmatrix}} \begin{matrix} B \\ \oplus \\ D \end{matrix} \xrightarrow{[f\ g]} E$$

where φ is an isomorphism. Then the complex

$$\cdots \to A \xrightarrow{e} C \xrightarrow{c - a\varphi^{-1}b} D \xrightarrow{g} E \to \cdots$$

is homotopy equivalent to the original.

Rmk: $B \xrightarrow{\sim} B$ is neither a sub, quotient, summand, but still it contracts out! ②

Ex:
$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} = B \\ \oplus & \oplus \\ B = \mathbb{C} \arrow[r, "1"] \arrow[uur, "1"] & \mathbb{C} \end{tikzcd}$$
contract at B and get
$$\begin{tikzcd} \mathbb{C} \arrow[dr, "-1"] & \\ & \mathbb{C} \end{tikzcd}$$
even though originally there was no map b/w these summands.

This is a really effective tool for simplifying complexes.

Pf Sketch: Step ① Show that the original complex is isomorphic to a complex

$$A \xrightarrow{\begin{bmatrix} a \\ e \end{bmatrix}} \begin{matrix} B \\ \oplus \\ C \end{matrix} \xrightarrow{\begin{bmatrix} \varphi & 0 \\ 0 & \underset{c-a\varphi^{-1}b}{*} \end{bmatrix}} \begin{matrix} B \\ \oplus \\ D \end{matrix} \xrightarrow{\begin{bmatrix} 0 & g \end{bmatrix}} E$$
, i.e.
$$\begin{matrix} A \to C \to D \to E \\ \oplus \\ B \xrightarrow{\sim} B \end{matrix}$$

this is "gaussian elimination", i.e. row reduction.

Step ② Contract out $B \xrightarrow{\sim} B$ w/ homotopy.

The scoop on cones

$$A^\bullet \xrightarrow{f} B^\bullet \rightsquigarrow \mathrm{Cone}(f) = \left(\begin{tikzcd} \arrow[r] & B^0 \arrow[r, "d_B"] & B^1 \arrow[r] & {} \\ & \oplus & & \\ \arrow[r] & A^1 \arrow[uu, "f"] \arrow[r, "-d_A"'] & A^2 \arrow[uu, "f"] & \end{tikzcd} \right)$$

Facts: ① $\oplus$ $0 \to B \xrightarrow{u} \mathrm{Cone}(f) \xrightarrow{v} A[1] \to 0$ a s.e.s. of complexes, w/ $vu = 0$.

② $$\mathrm{Cone}(u) = \left(\begin{tikzcd} B^1 \arrow[r, "-d"] \arrow[dr, "id"] & B^2 \arrow[r] \arrow[dr, "id"] & {} \\ B^0 \arrow[r, "d"] & B^1 \arrow[r, "d"] & {} \\ A^1 \arrow[u, "f"] \arrow[r, "-d"'] & A^2 \arrow[u] \arrow[r] & {} \end{tikzcd} \right) \underset{G.E.}{\simeq} A[1]$$

③ $$\mathrm{Cone}(v) = \left(\begin{tikzcd} B^1 \arrow[r, "-d"] & B^2 \\ A^2 \arrow[u, "f"] \arrow[r, "d"] \arrow[dr, "id"] & A^3 \arrow[dr, "id"] \\ A^1 \arrow[r, "-d"'] & A^2 \arrow[r] & {} \end{tikzcd} \right) \simeq B[1].$$
So $\mathrm{Cone}(v)[-1] \cong \mathrm{Cyl}(f)$ is homotopic to B.

④ Composition $A \xrightarrow{f} B \xrightarrow{u} \mathrm{Cone}(f)$ is nonzero, but is nullhomotopic

$C(f) \to A[1] \xrightarrow{f[1]} B[1]$

(5) Applying (i) to Cone(v) Cone(u) get more s.e.s

(**) $0 \to A \to Cyl(f) \to Cone(f) \to 0$

(***) $0 \to Cone(f) \to \underset{}{\overset{\text{is } A[1]}{Cone(u)}} \to B[1] \to 0$

(6) In l.e.s. of (*)

$$
\begin{tikzcd}
h^k(B) \arrow[r] & h^k(Cone(f)) \arrow[r] & h^k(A[1]) \arrow[r, "\delta"] & h^{k+1}(B) \\
& & h^{k+1}(A) \arrow[u, equal] \arrow[ur, "f_*"'] &
\end{tikzcd}
$$

← this triangle commutes. Was an exercise.

(7) In l.e.s. of (**)

$$
\begin{tikzcd}
h^k(A) \arrow[r] \arrow[dr, "f_*"'] & h^k(Cyl(f)) \arrow[d, "\text{IIS}"] \\
& h^k(B)
\end{tikzcd}
$$

is also f_*.

putting this together: the l.e.s. of $0 \to B \to Cone(f) \to A[1] \to 0$

is $\dots \to h^k(A) \to h^k(B) \to h^k(Cone(f)) \to h^{k+1}(A) \to \dots$

is the same as the l.e.s of

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & Cyl(f) \arrow[r] \arrow[d, "\text{IIS}"] & Cone(f) \arrow[r] & 0 \\
& & B & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & Cone(f) \arrow[r] & Cone(u) \arrow[r] \arrow[d, "\text{IIS}"] & B[1] \arrow[r] & 0 \\
& & A[1] & &
\end{tikzcd}
$$

We think of these s.e.s as "rotations" of each other — cycle the three terms, but then replace them w/ hom. eq. things to get a s.e.s of complexes

(Can't just cycle naively: $A \xrightarrow{f} B \xrightarrow{u} Cone(f)$ composition is not even zero, f not injective not surjective etc etc.)

This is the key idea, but first some major warnings about cones.

(8) Cones are NOT "canonical". What are the maps

$$
\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & Cone(f) \arrow[r] \arrow[d, "\varphi", "S"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & Cone(f) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}
$$

?

$$\varphi = \begin{pmatrix} id_B & h \\ 0 & id_A \end{pmatrix}$$

to be a chain map compute

$$dh + hd = 0 \quad , \quad -dh = hd$$

$h : A^1 \to B^0$
$A^2 \to B^1$
$\vdots$

i.e. h is a chain map $A \to B[-1]$.

$\varphi \simeq \mathrm{id}_{\mathrm{Cone}} \iff h \simeq 0$ as maps $A \to B[-1]$,

so even in the homotopy category, a non-nullhomotopic map $A \to B[-1]$ (think: an element of $\mathrm{Ext}^{-1}(A^\circ, B^\circ)$) gives a non-trivial automorphism of $\mathrm{Cone}(f)$.

(9) It is obvious that a) f q.isom $\Rightarrow$ $\mathrm{Cone}(f)$ exact from l.e.s.

b) f isom $\Rightarrow$ $\mathrm{Cone}(f)$ contractible by G.E.

but if f is h.e., it need not be the case that $\mathrm{Cone}(f)$ is contractible!!

__Exercise:__ A nullhomotopy for $\mathrm{id}_{\mathrm{Cone}(f)}$ is the same data as: $h = \begin{pmatrix} g & m \\ k & n \end{pmatrix}$

- a homotopy inverse k for f
- a homotopy g b/w fk and 1
- — " — $-n$ b/w kf and 1

so gf, $-fn$ are both homotopies for fkf.

$\Rightarrow gf + fn$ is a homotopy for zero, i.e. a closed map $A \to B[-1]$

- — " — m b/w $gf + fn$ and 0. (need not exist!!)

Cones are great... but not perfect!! This causes some technical issues, but not too many.