Triang Cats | Recall some defns from day 1.

Def: $K(\mathcal{A})$ the homotopy category

$\mathcal{A}$ an additive (or abelian) cat.

Ob: same as $Ch(\mathcal{A})$

Mor: $\mathrm{Hom}_{K(\mathcal{A})}(A,B) = \mathrm{Hom}_{Ch(\mathcal{A})}(A,B) /$ nullhomotopy maps

It is a graded cat, equipped w/ invertible grading shift functor (or translation functor) $[1]$. Here, $A[1]^i = A^{i+1}$ and $d_{A[1]} = -d_A$

$A \xrightarrow{f} B$ then $A[1] \xrightarrow{f[1]} B[1]$ is the same underlying map (no sign).

Prop: In $K(A)$, any h.e. is an isom. Moreover, the quotient functor $Ch(\mathcal{A}) \longrightarrow K(\mathcal{A})$ is universal wrt this property. Pf: See Weibel, fun cylinder exercise.

Recall: $K(A)$ is still additive, but even when A is abelian, $K(A)$ is rarely abelian.

$0 \to B \xrightarrow{u} \mathrm{Cone}(f) \xrightarrow{v} A[1] \to 0$ $\quad$ $u = \mathrm{Ker}\, v$ but u is not monic

$A \xrightarrow{f} B \xrightarrow{u} \mathrm{Cone}(f)$ $\quad$ $uf \simeq 0$ but $f \not\simeq 0$

Instead $K(A)$ is triangulated.

Def: Let K be an additive graded cat. A triangle in K is the data

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

which also leads to the data $\ldots \to C[-1] \xrightarrow{h[-1]} A \to B \to C \to A[1] \xrightarrow{f[1]} B[1] \xrightarrow{g[1]} C[1] \xrightarrow{h[1]} A[2] \to \ldots$

A morphism of triangles is

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & A[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \to & Y & \to & Z & \to & X[1] \end{array}$$

all squares commute.

Def: A triangulated cat is an additive graded cat K equipped with a collection Δ of special triangles called distinguished triangles, satisfying axioms TR0–TR4 below.

To motivate these axioms:

Thm: $K(A)$ is triangulated, when $\Delta = \left\{ \begin{array}{l} \text{triangles isomorphic in } K(\mathcal{A}) \text{ to} \\ A \xrightarrow{f} B \xrightarrow{u} \mathrm{Cone}(f) \xrightarrow{v} A[1] \\ \text{for some chain map } f \end{array} \right\}$

think: the Δ comes from the ses $0 \to B \to \mathrm{Cone}(f) \to A[1] \to 0$

Remark:
$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$
is a s.e.s. of complexes,

but does NOT give rise to a d.t. in $K(\mathbb{Z}\text{-mod})$!

only degreewise-split s.e.s. do!! Recall our exercise: for every s.e.s

$0 \to P \xrightarrow[\text{incl}]{} X \xrightarrow[\text{proj}]{} Q \to 0$ where $X^i = P^i \oplus Q^i$, X is a Cone of $P \to Q[-1]$.

Axioms: (TR0) 1) ~~[illegible]~~

Δ is closed under $\cong$ of triangles.

2) $(A \xrightarrow{id} A \to 0 \to A[1]) \in \Delta$ for all $A \in Ob(K)$

Pf for $K(A)$: ① by defn. ② b/c $Cone(id) \simeq 0$

(TR1) Any map $A \xrightarrow{f} B$ extends to a ~~[illegible]~~ D.T. $(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]) \in \Delta$.

Rmk: Other axioms will ensure C is well defined up to iso, but NOT up to unique iso! Same problem we noticed for cones earlier. In general, C is called a ~~[illegible]~~ Cone of f.

Pf for $K(A)$: Take the cone.

With ~~[illegible]~~ the next axiom, an erratum:

(TR2) given the triangle $\Delta = (A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1])$ define

$$rot(\Delta) = (B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1])$$ W/ a sign!

and $$rot^{-1}(\Delta) = (C[-1] \xrightarrow{-h[-1]} A \xrightarrow{f} B \xrightarrow{g} C).$$ Then $\Delta \in \Delta \Rightarrow rot^{\pm}(\Delta) \in \Delta$.

Pf for $K(A)$, w/ erratum: WTS

$$\begin{array}{ccccccc} B & \xrightarrow{u} & Cone(f) & \xrightarrow{v} & A[1] & \xrightarrow{-f[1]} & B[1] \\ \| & & \| & & \downarrow\uparrow & & \| \\ B & \xrightarrow{u} & Cone(f) & \xrightarrow{i} & Cone(u) & \xrightarrow{\epsilon} & B[1] \end{array}$$
is in Δ, i.e. $\cong$ to ←

(on board, flip upside-down)

$$Cone(u) = \left(\begin{array}{ccccc} B^1 & \xrightarrow{-d} & B^2 & \xrightarrow{-d} & \\ & \searrow^{id} & & \searrow^{id} & \\ B^0 & \xrightarrow{d} & B^1 & \xrightarrow{d} & \\ & \nearrow^{f} & & \nearrow^{f} & \\ A^1 & \to & A^2 & \to & \end{array} \right) \underset{G.G.}{\simeq} A[1]$$
but the map $A[1] \to Cone(u)$ is NOT "inclusion", not a chain map.

We discussed how S.G. affects differentials but not chain maps.

The actual chain map $A[1] \to \mathrm{Cone}(u)$ is $a \mapsto (a, a - f(a))$

Thus $A[1] \to \mathrm{Cone}(u) \xrightarrow{q} B[1]$ is $-f[1]$, not $+f[1]$!!

But then $\mathrm{Cone}(f) \xrightarrow{v} A[1] \to \mathrm{Cone}(u)$ is not p! ~~Need to~~

But it is homotopic to p!! Whew.

(TR3) Given

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\exists\gamma"] & A[1] \arrow[d, "\alpha[1]"] & \in \Delta \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] & \in \Delta \end{tikzcd}$$

$\exists \gamma$, (but not necc. unique!)

Pf for $K(A)$: Easy exercise w/ cones.

Before getting to the crazy axiom (TR4), some consequences.

⓪
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "\cong -1"] & C \arrow[r, "h"] \arrow[d, equal] & A[1] \arrow[d, equal] \\ A \arrow[r, "-f"] & B \arrow[r, "-g"] & C \arrow[r, "h"] & A[1] \end{tikzcd}$$
so $-f, -g, h$ also dist.

① $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \in \Delta \Rightarrow gf = 0$

$\left(\underset{\text{rot}}{\Longrightarrow} hg = 0,\ f[1] \circ h = 0 \right)$

Pf:

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, "\exists"] & A \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & A \end{tikzcd}$$

② Def: A functor $\underset{\text{tri}}{K} \xrightarrow{h} \underset{\text{abelian}}{\mathcal{B}}$ is a homological functor iff

$\ldots \to h(C[-1]) \to h(A) \to h(B) \to h(C) \to h(A[1]) \to h(B[1]) \to \ldots$ is exact.

Prop: ~~XXXX~~ If $\mathcal{A}$ abelian, $h^0 : K(\mathcal{A}) \to \mathcal{A}$ is a homological functor

Rmk: $\mathcal{A}$ additive then might not have built-in hom functor! on $K(\mathcal{A})$.

Thm: For any $X \in \mathrm{Ob}(K)$, $\mathrm{Hom}_K(X, -)$ is a homological functor to $\mathbb{Z}$-mod.

Pf: $\exists$ TS $\mathrm{Hom}(X,A) \xrightarrow{f_\circ} \mathrm{Hom}(X,B) \xrightarrow{g_\circ} \mathrm{Hom}(X,C)$ exact in middle, then rotate to get exact seqn.

- composition is zero by ①
- if $\varphi : X \to B$ and $g\varphi = 0$ then $\exists \tilde{\varphi} : X \to A$ as follows

$$\begin{tikzcd} X \arrow[r] \arrow[d, dashed, "\exists \tilde\varphi"'] & X \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, dashed, "\exists"] \\ A \arrow[r] & B \arrow[r, "i"] & C \arrow[r] & A[1] \end{tikzcd}$$

(thus for TR3 for the rotated triangle)

so φ is $f \circ \tilde\varphi$. ■

③ Many corollaries

Cor (5-lemma)
$$\begin{tikzcd} X \arrow[r] \arrow[d, "\wr"'] & Y \arrow[r] \arrow[d, "\wr"] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, "\wr"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd} \Longrightarrow \begin{tikzcd} Z \arrow[d, "\wr"] \\ Z' \end{tikzcd}$$

Cor: $X \xrightarrow{\sim} Y$ then $Z \cong 0$

Cor $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ then $w = 0 \Longleftrightarrow$ triangle is isomorphic to $\left(X \xrightarrow{inc} X \oplus Z \xrightarrow{proj} Z \xrightarrow{0} X[1] \right)$

and such triangles are always distinguished.

Now for TR4, 3rd isom theorem: $C \overset{i}{\subset} B \overset{j}{\subset} A$ modules then $A/B \cong (A/C)/(B/C)$

$$\begin{tikzcd} & & & 0 \\ 0 \arrow[r] & B \arrow[r, "j"] & A \arrow[r] & A/B \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & C \arrow[r, "ji"] & A \arrow[r] & A/C \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & C \arrow[r, "i"] & B \arrow[r] & B/C \arrow[r] \arrow[u] & 0 \\ & & & 0 \arrow[u] \end{tikzcd}$$

← think of these as d.t.

(TR4) Given $X \xrightarrow{\alpha} Y$ and $Y \xrightarrow{\beta} Z$ (not part of one triangle)

then $\exists$ d.t. $C_\alpha \xrightarrow{\varphi} C_{\beta\alpha} \xrightarrow{\psi} C_\beta \xrightarrow{\gamma} C_\alpha[1]$ (where C_α is any cone of α, w/ fixed triangle.)

s.t. ⓐ
$$\begin{tikzcd} & X[1] \arrow[r, equal] & X[1] \\ C_\beta[-1] \arrow[r, "-\gamma[-1]"] & C_\alpha \arrow[r, "\varphi"] \arrow[u] & C_{\beta\alpha} \arrow[r, "\psi"] \arrow[u] & C_\beta \\ C_\beta[-1] \arrow[r, "-"] \arrow[u, equal] & Y \arrow[r, "\beta"] \arrow[u] & Z \arrow[r] \arrow[u] & C_\beta \arrow[u, equal] \\ & X \arrow[r, equal] \arrow[u, "\alpha"] & X \arrow[u, "\beta\alpha"'] \end{tikzcd}$$

b/c of rot

AND ⓑ
$$\begin{tikzcd} & Z[-1] \arrow[r, equal] \arrow[d] & Z[-1] \arrow[d, "-"] \\ C_\alpha & C_\beta[-1] \arrow[l, "-\gamma[-1]"'] \arrow[d, "-"] & C_{\beta\alpha}[-1] \arrow[l, "\psi[-1]"'] \arrow[d, "-"] & C_\alpha[-1] \arrow[l, "\varphi[-1]"'] \\ C_\alpha \arrow[u, equal] & Y \arrow[l] \arrow[d, "\beta"'] & X \arrow[l, "\alpha"'] \arrow[d, "\beta\alpha"] & C_\alpha[-1] \arrow[l, "-"'] \arrow[u, equal] \\ & Z \arrow[r, equal] & Z \end{tikzcd}$$

this is rot^3 followed by double negation

"outside square of ⓐ is inside square of ⓑ but with a sign!!

i.e. outside squares ANTICOMMUTE!!!

the rest are triangles. Try gluing in 3D!

octahedron axiom

octaeder

Yikes!! Only way to appreciate (TR4) is to use it. Really necessary when proving facts about t-structures (soon). OR just move on w/ your life.

Rmk: Two main examples of tri. cat.: ① $K(\mathcal{A})$ and K^+, K^-, K^b

② Stable modules over a Frobenius algebra A — Ob: A-mod; Mor: A-mod maps / morphisms factoring thru projectives (= injectives)

Think: Killing projectives is like killing contractible complexes (exercise).

② is great b/c a good way to appreciate axioms is to use them in an unfamiliar context!!

③ Derived categories. Next in line.

Aside: The triangulated Grothendieck gp $[K]$ is $\mathbb{Z}\langle [M] \mid M \in \mathrm{Ob}(K)\rangle \big/ [A]+[C]=[B]$ when $A \to B \to C \to A[1]$ is d.t.

Note: $[A[1]] = -[A]$ since $A \to 0 \to A[1]$ is d.t.

Exercise: $[K^b(\mathrm{Vect}_{f.d.})] \cong \mathbb{Z}$. $\quad C^\bullet \longmapsto \chi(C^\bullet)$

Hint: Build a complex as an iterated cone, degree by degree.

Def: Let $\mathcal{A}$ be an abelian cat. The derived cat of $\mathcal{A}$, $D(\mathcal{A})$, is obtained from $K(\mathcal{A})$ by inverting all qisoms.

(Rmk: When $\mathcal{A}$ additive, no notion of qisom, no derived cat.)

Two things need to be done, quite separate: ① make sense of this ② how to use it

Preview of ②: Any $M \in \mathcal{A} \rightsquigarrow 0 \to M \to 0$ in $K(\mathcal{A})$ is qisom to a proj resolution $P^\bullet$

In fact, any bdd above complex $M^\bullet$ is qisom to a complex of projectives, called the Cartan–Eilenberg res'n (soon). So this is the correct formal context for working w/ proj res'ns.

$$\underset{\text{bdd above}}{D^-(\mathcal{A})} \cong \underset{\text{just additive}}{\text{“}D^-(\mathrm{Proj}\,\mathcal{A})\text{”}} \cong K^-(\mathrm{Proj}\,\mathcal{A})$$

any qisom b/w complexes of projectives is a h.e.

So to compute morphisms in $D^-(A)$, replace all complexes w/ projective replacements and then ~~write~~ find chain maps up to homotopy. More later.

Let's get ① out of the way. What does "inverting qisoms" really mean?

Motivation: R a comm. ring. $S \subset R$ a mult. set: ⓪ $1 \in S$ ① $f, g \in S \Rightarrow fg \in S$

Then $R[S^{-1}]$ is well defined: $R[S^{-1}] = \left\{ \underset{\overset{\|}{af^{-1}}}{\frac{a}{f}} \;\middle|\; a \in R, f \in S \right\} \Big/ \frac{a}{f} \sim \frac{b}{g} \Leftrightarrow \exists h \in S$ s.t. $h(ag - bf) = 0$

(equiv, $\frac{a}{f} = 0 \Leftrightarrow \exists s \in S$ s.t. $as = 0$)

What if R is a non-comm ring?! Two problems w/ defining $R[S^{-1}]$.

a) Need elements like $af^{-1}bg^{-1}ch^{-1}\dots$ making $R[S^{-1}]$ too big!

b) $a = 0$ if $\exists s \in S$ s.t. $sa = 0$
OR if $\exists t \in S$ s.t. $at = 0$
or ______ $sat = 0$

but these conditions are not equivalent.

Def: A collection of morphisms S in a category $\mathcal{C}$ is a localizing class if

⓪ $1_X \in S$ $\forall X$ ① $f, g \in S \Rightarrow fg \in S$ (composable) ② For all
$$\begin{tikzcd} & X \arrow[d, Rightarrow, "f\in S"] \\ Z \arrow[r, "a"'] & Y \end{tikzcd} \quad \exists \quad \begin{tikzcd} W \arrow[r, "b"] \arrow[d, Rightarrow, "s \ni g"'] & X \arrow[d, Rightarrow, "f"] \\ Z \arrow[r, "a"'] & Y \end{tikzcd}$$
and the other way around too!

③ $\forall a$, $\exists s \in S$ s.t. $sa = 0$ $\Updownarrow$ $\exists t \in S$ s.t. $at = 0$.

Use double edges to denote a morphism in S

② is called Ore condition. Says $\forall a, f\ \exists g, b$ s.t. $af^{-1} = g^{-1}b$ $f^{-1}a = bg^{-1}$. Hence no long words $af^{-1}bg^{-1}\dots$ needed, words like af^{-1} will span.

Def: If S a localizing class, define $\mathcal{C}[S^{-1}]$ with $Ob = Ob(\mathcal{C})$

$$\mathrm{Hom}(X, Y) = \left\{ \begin{tikzcd} & Z \arrow[dl, Rightarrow, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd} \right\} \Big/ \sim$$

we call these roofs.

Composition:

$$\begin{tikzcd}[column sep=small] & & U \arrow[dl, Rightarrow] \arrow[dr] & & \\ & Z \arrow[dl, Rightarrow] \arrow[dr] & & W \arrow[dl, Rightarrow] \arrow[dr] & \\ X & & Y & & V \end{tikzcd}$$

compose to

$$\begin{tikzcd}[column sep=small] & U \arrow[dl, Rightarrow] \arrow[dr] & \\ X & & V \end{tikzcd}$$

indep of choice up to equivalence.

equiv:

$$\begin{tikzcd} & Z \arrow[dl, Rightarrow] \arrow[dr] & \\ X & Z'' \arrow[l, Rightarrow] \arrow[r] \arrow[u] \arrow[d] & Y \\ & Z' \arrow[ul, Rightarrow] \arrow[ur] & \end{tikzcd}$$

Confirm that:

- $$\begin{tikzcd}[column sep=small] & Z \arrow[dl, Rightarrow, "s"'] \arrow[dr, "s"] & \\ X & & X \end{tikzcd} \sim \begin{tikzcd}[column sep=small] & X \arrow[dl, Rightarrow, "id"'] \arrow[dr, "id"] & \\ X & & X \end{tikzcd}$$

- this is $\begin{tikzcd}[column sep=small] & Z \arrow[dl, Rightarrow, "s"'] \arrow[dr, Rightarrow, "id"] & \\ X & & Z \end{tikzcd}$ composed w/ $\begin{tikzcd}[column sep=small] & Z \arrow[dl, Rightarrow, "id"'] \arrow[dr, "s"] & \\ Z & & X \end{tikzcd}$

- $f \longmapsto$ $\begin{tikzcd}[column sep=small] & X \arrow[dl, Rightarrow, "id"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$ gives functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$
  $x \to y$

Rmk: Is $\{\text{roofs}\}/\sim$ even a set? Roofs thru fixed object Z is, but $Ob(\mathcal{C})$ isn't.

Often people add one more condition on localizing classes, $\exists$ set of objects where all roofs go thru these up to equiv. Locally Small

Thm (Gabriel–Zisman): Size of Homs in $\mathcal{C}[S^{-1}]$ is as expected.

$\mathcal{C} \to \mathcal{C}[S^{-1}]$ is universal amongst functors where $S \to$ Isoms.

~~Inside $K(A)$, qIsoms form a localizing class~~ ~~DE: lots of work, not deep~~

go to next page

Big Technical Remark:

$$\begin{tikzcd} Ch(A) \arrow[r, "Q"] \arrow[rr, bend right, "\Phi"'] & K(A) \arrow[r, "i"] & D(A) \end{tikzcd}$$

Q universal h.e. $\to$ isom

i $\leftarrow$ qisom $\to$ isom

it turns out also that Φ is universal qisoms $\to$ isoms

So why not just $D(A) = Ch(A)[\text{qisoms}^{-1}]$? B/c qisom NOT a localizing class inside $Ch(A)$!!!

But ultimately, $\mathrm{Hom}_{D(A)}(A^\circ, B^\circ) =$ Some crazy roofs mod equivalence

You NEVER use this. Next step is to understand this better.

Now for a key application:

<u>Thm:</u> Let K be triangulated, $h: K \to \mathcal{B}$ a homological functor, and $Q = \{ f \mid h(f) \text{ is an isom} \}$. Then Q is a localizing class and $K[Q^{-1}]$ is triangulated.

<u>Sketch:</u>

- Q is multiplicative $\boxed{\checkmark}$
- given

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "g"'] & Y \arrow[r, "u \circ f"] \arrow[d, "f"] & C \arrow[r, dashed] \arrow[d, equal] & X[1] \arrow[d] \\ A \arrow[r, Rightarrow, "s"] & B \arrow[r, dotted, "u"] & C \arrow[r] & A[1] \end{tikzcd}$$

WTS $t \in Q$

apply h. Since $h(s)$ is isom, $h(s[1])$ is isom, so l.e.s. is

$$\cdots\ h(A) \xrightarrow{\sim} h(B) \to h(C) \to h(A[1]) \xrightarrow{\sim} h(B[1]) \to \cdots$$

$$\Rightarrow h(C) = 0$$

$\Rightarrow$ l.e.s. is $0 \to h(X) \xrightarrow{h(t)} h(Y) \to 0 \to \cdots \Rightarrow t \in Q$.

- <u>Exercise:</u> $\exists s,\ sf = 0 \iff \exists t,\ ft = 0$.

- Define a triangle in $K[Q^{-1}]$ as the image of DT from K

(Warning: major! erratum in Weibel!)

<u>Ex:</u> What is $\mathrm{Cone}\left(A \overset{s}{\Leftarrow} X \xrightarrow{f} B\right)$? $\exists$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, Rightarrow, "s"'] & B \arrow[r, "g"] & C \arrow[r, "h"] & X[1] \arrow[d, Rightarrow, "s[1]"] \\ A & & & A[1] \end{tikzcd}$$

so let it be

$$A \overset{s}{\Leftarrow} X \to B \overset{1}{\Leftarrow} B \to C \overset{1}{\Leftarrow} C \xrightarrow{s[1]h} A[1]$$