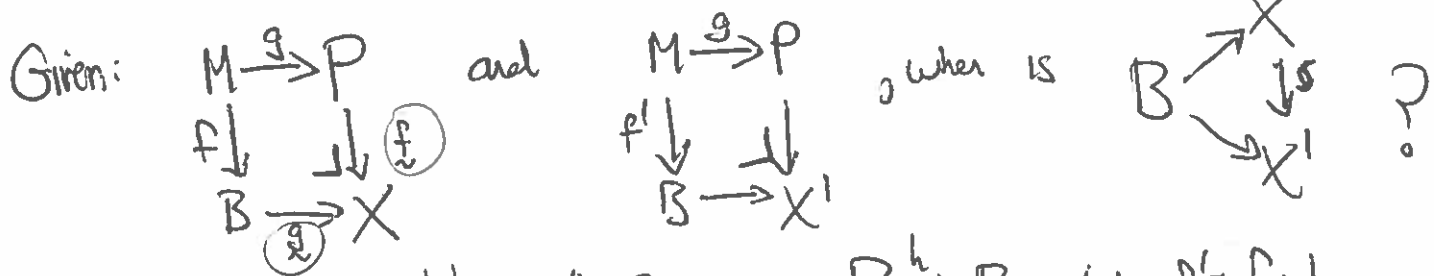


# Yoneda Ext | ? I gotta ext. | Why ext is called ext.

①

First: more on pushouts.



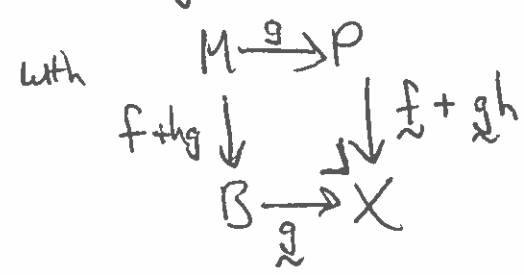
Here's a situation where it happens: suppose  $P \xrightarrow{h} B$ . Let  $f' = f + hg$ .

$$X = \text{Coeq}(M \rightrightarrows B \oplus P) = B \oplus P / \langle (f(m), -g(m)) \rangle = I$$

$$X' = B \oplus P / \langle (f(m) + hg(m), -g(m)) \rangle = I'$$

$B \oplus P$  has autom.  $\begin{pmatrix} 1_B & -h \\ 0 & 1_P \end{pmatrix}$  which preserves  $B$  and sends  $\begin{pmatrix} b \\ p \end{pmatrix} \mapsto \begin{pmatrix} b-hp \\ p \end{pmatrix}$   
 so sends  $I$  to  $I'$ . Indeed  $X \cong X'$ .

Send another way,  $X$  would also serve as the pushout of diagram 2,



(sure looks like a homotopy to me!)

$$\begin{array}{c} \text{Pushouts} \\ M \xrightarrow{g} P \\ \downarrow \\ B \end{array} / \cong \text{rel } B \quad \longleftarrow \quad \text{Hom}(M, B) / \text{Hom}(P, B) \circ g$$

This is isom, one must check.

Def: An extension of  $A$  by  $B$  is

$$0 \rightarrow B \xrightarrow{f} X \xrightarrow{g} A \rightarrow 0$$

(2)

equivalent if

$$\begin{array}{ccccccc} 0 & \rightarrow & B & \xrightarrow{f} & X & \xrightarrow{g} & A \rightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \rightarrow & B & \xrightarrow{f'} & X' & \xrightarrow{g'} & A \rightarrow 0 \end{array}$$

(note: It's an isom, but let's view that as an accident...)

spltd if equiv to  $0 \rightarrow B \rightarrow B \oplus A \rightarrow A \rightarrow 0$ .

$$\text{Ext}^1(A, B) := \text{Extensions} / \cong$$

Thm (Yoneda): 1)  $\text{Ext}^1(A, B)$  is an abelian group (as below)!  
 Baer 2)  $\text{Ext}^1(A, B) \cong \text{Ext}^1(A, B)$  as ab groups functorially in  $A$  and  $B$ .

First we discuss the bijection.

Given  $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$  ~~then~~ apply  $\text{Hom}(A, -)$  to get less

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(A, B) & \rightarrow & \text{Hom}(A, X) & \rightarrow & \text{Hom}(A, A) \xrightarrow{\delta_A} \text{Ext}^1(A, B) \rightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1_A & \mapsto & \delta_A(1_A) & & \end{array}$$

Functionality of here says indep of equiv class

OR apply  $\text{Hom}(-, B)$  to get

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(A, B) & \rightarrow & \text{Hom}(X, B) & \rightarrow & \text{Hom}(B, B) \xrightarrow{\delta_B} \text{Ext}^1(A, B) \rightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 1_B & \mapsto & \delta_B(1_B) & & \end{array}$$

Under future identification of two versions of  $\text{Ext}^1(A, B)$ ,  $\delta_A(1_A)$  matches  $\delta_B(1_B)$ .

Another view:  $(P^\bullet) \Rightarrow A$  then comp. lemma gives

$$\begin{array}{ccccccc} \dots & \rightarrow & P_{-1} & \rightarrow & P_0 & \rightarrow & A \rightarrow 0 \\ & & \downarrow x & & \downarrow & & \downarrow \\ 0 & \rightarrow & B & \rightarrow & X & \rightarrow & A \rightarrow 0 \end{array}$$

It is  $X \in \text{Hom}(P_{-1}, B)$  which gives a class in  $\text{Hom}(P^\bullet, B)$  descending to  $\delta_B(1_B)$  in  $H^1(\text{Hom}(P^\bullet, B)) = \text{Ext}^1(A, B)$

Another view:  $0 \rightarrow M \xrightarrow{g} P \rightarrow A \rightarrow 0$  w/  $P$  projective.

For any ~~kernel~~ ~~kernel~~ ~~kernel~~ right exact  $F$  we have

$$\dots \rightarrow \tilde{L}FA \rightarrow \tilde{L}FM \rightarrow 0 \rightarrow \tilde{L}FA \rightarrow FM \rightarrow FP \rightarrow FA \rightarrow 0$$

dimensional reduction.

So  $\text{Ext}^1(A, B) \cong \text{Hom}(M, B) / \text{Hom}(P, B)_{\text{og}}$

Given  $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$  get

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & P & \rightarrow & A \rightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X & \rightarrow & A \rightarrow 0 \end{array}$$

and  $\text{Hom}(M, B)$  descends to  $S_B(1_B)$   
"  $\text{Ext}^1(A, B)$

★ GO TO (3) ★

Conversely, given  $f \in \text{Hom}(M, B)$  can construct pushout

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \xrightarrow{g} & P & \rightarrow & A \rightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X & \rightarrow & A \rightarrow 0 \end{array}$$

b/c pushout preserves cokernel.

this yields an extension. As noted, gives a map  $\text{Ext}^1(A, B) = \text{Hom}(M, B) / \text{Hom}(P, B)_{\text{og}}$  to  $\text{Ext}^1(A, B)$

Claim: These maps are inverse, i.e. if

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \xrightarrow{f} & P & \rightarrow & A \rightarrow 0 \\ & & \downarrow g & & \downarrow & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X & \rightarrow & A \rightarrow 0 \end{array}$$

then  $X \cong$  pushout. (rel  $B$ )

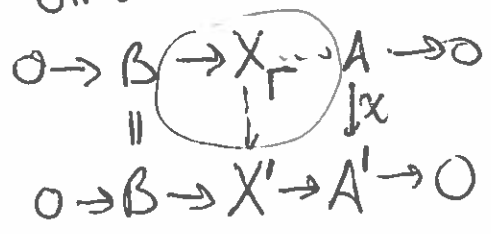
Pf: Have map  $B \oplus P \xrightarrow{g} X$  and kernel contains  $\langle (g(m), -f(m)) \rangle$

I'll use elements to show isom.

- $q \circ j$  injective:  $f: X \rightarrow X, x \mapsto \alpha x$  then  $\exists p \in P, p \mapsto \alpha$   
 $x - j(p) \mapsto 0$  so  $\exists b \in B, b \mapsto x - j(p)$ .
- if  $(b, p) \mapsto 0 \in X$  then  $q(b) = j(p) \Rightarrow j(p) \mapsto 0 \in A$   
 $\Rightarrow p \mapsto 0 \in P$   
 $\Rightarrow \exists \text{MGM}, m \mapsto p$ . B/c  $B \rightarrow X$  is injective,  $\begin{matrix} m \\ \downarrow \\ 0 \end{matrix}$ .

We have the bijection. How is Ext's functorial??

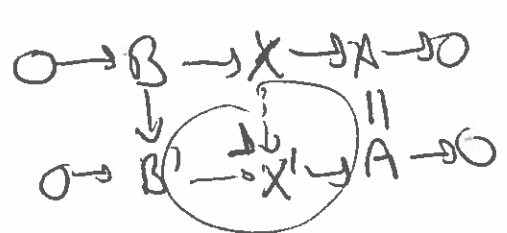
Given  $A \xrightarrow{\alpha} A'$  want  $\text{Ext}^1(A', B) \rightarrow \text{Ext}^1(A, B)$



use pullback square

Functoriality of  $\text{Ext}$  says this commutes with map to  $\text{Ext}^1(-, B)$

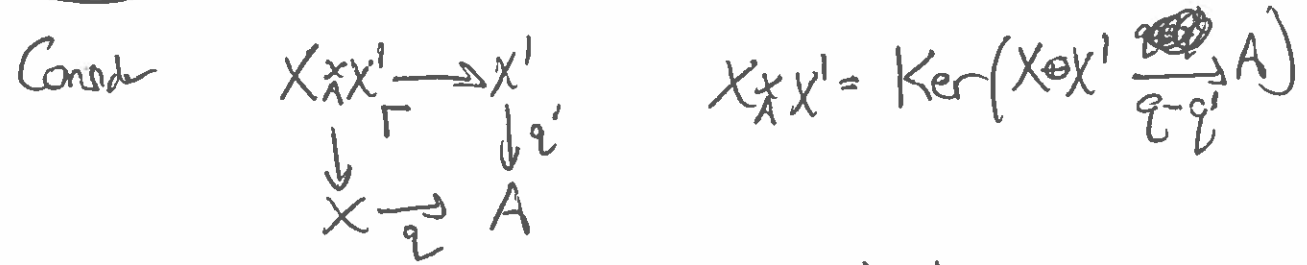
Given  $B \xrightarrow{\beta} B'$  want  $\text{Ext}^1(A, B) \rightarrow \text{Ext}^1(A, B')$



Use pushout.

How is  $\text{Ext}^1(A, B)$  an abelian group???

Baer Sum: Given  $0 \rightarrow B \xrightarrow{\alpha} X \xrightarrow{\beta} A \rightarrow 0$  and  $0 \rightarrow B' \xrightarrow{\alpha'} X' \xrightarrow{\beta'} A \rightarrow 0$



Now  $B$  actually has lots of maps to  $X \times X'$ , two in particular.

$B \rightarrow X \times X'$  and  $B \rightarrow X \times X'$ , let's make them equal.  
 $b \mapsto (0(b), 0)$  and  $b \mapsto (0, q'(b))$

Let  $Y = X \times_A X' / \langle (g(b), 0) - g'(b) \rangle$  i.e.  $(g(b), 0) = (0, g'(b))$  in quotient. (5)

Then  $0 \rightarrow B \rightarrow Y \rightarrow A \rightarrow 0$  is exact (exercise).  
 with  $\uparrow$   $b \mapsto (g(b), 0)$  or  $(0, g'(b))$

$Y$  is the Baer Sum of  $X$  and  $X'$ .

Comparison w/  $\text{Ext}^1$ : Fix  $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$  and consider

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & P & \rightarrow & A \rightarrow 0 \\ & & \downarrow f & & \downarrow f' & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X & \rightarrow & A \rightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & P & \rightarrow & A \rightarrow 0 \\ & & \downarrow f' & & \downarrow f & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X' & \rightarrow & A \rightarrow 0 \end{array}$$

The  $P \rightarrow X \oplus X'$  has image in  $X \times_A X'$ , so get map  $P \rightarrow Y$   
 $\downarrow$   $\cdot j \circ j'$

Exercise:

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & P & \rightarrow & A \rightarrow 0 \\ & & \downarrow f & & \downarrow f' & & \parallel \\ 0 & \rightarrow & B & \rightarrow & Y & \rightarrow & A \rightarrow 0 \end{array}$$

$f \circ f'$  is induced map.

Identity is the split extension!

1) For  $\text{Ext}^1$ ,  $\mathcal{S} = 0$  for split seq (Why?) (i.e. splits  $0 \rightarrow B \rightarrow B \rightarrow 0$  over  $0 \rightarrow A \rightarrow A \rightarrow 0$ )

2) For Baer,  $X \times_A (B \oplus A) \cong X \oplus B$   
 $Y = X \oplus B / \langle (g(b), 0) - (0, b) \rangle \cong X$

★ Go to (5) ★



Great... what about  $\text{Ext}^2(A, B)$ ??

(6)

Def:  $\text{Ext}^2(A, B) = \{ 0 \rightarrow B \rightarrow X_1 \xrightarrow{\text{exact}} X_2 \rightarrow A \rightarrow 0 \} / \text{equiv}$

where: 
$$\begin{array}{ccccccc} 0 & \rightarrow & B & \rightarrow & X_1 & \rightarrow & X_2 \rightarrow A \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X_1' & \rightarrow & X_2' & \rightarrow & A \rightarrow 0 \end{array}$$
 (not nec isoms!!)

Primitively equiv if

This isn't equiv reln, so take equiv reln generated by this.

Similarly,  $\text{Ext}^k(A, B) = \{ 0 \rightarrow B \rightarrow X_1 \rightarrow \dots \rightarrow X_k \rightarrow A \rightarrow 0 \} / \text{equiv}$

Thm (Y):  $\text{Ext}^k(A, B) \cong \text{Ext}^k(A, B)$  as abelian grp, functorially

How? Choose  $0 \rightarrow M \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$

$$\begin{array}{ccccccc} & & \downarrow & \downarrow & \downarrow & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X_1 & \rightarrow & X_2 \rightarrow A \rightarrow 0 \end{array}$$

dimensional reduction:  
 $\text{Ext}^2(X, B) \cong \text{Hom}(M, B) / \text{Hom}(P_1, B)$

exists by comparison lemma (the map  $M \rightarrow B$  induced via kernels)

Conversely, given  $f \in \text{Hom}(M, B)$  get

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & P_1 & \rightarrow & P_0 \rightarrow A \rightarrow 0 \\ & & f \downarrow & & \downarrow & & \parallel & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X & \rightarrow & P_0 & \rightarrow & A \rightarrow 0 \end{array}$$

for injection, check: exists

$$\begin{array}{ccccccc} & & \parallel & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & B & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow & A \rightarrow 0 \end{array}$$

Pf: Ginnormous pushback exercise...

Note:  $0 \rightarrow B \rightarrow X_1 \rightarrow \dots \rightarrow X_k \rightarrow A \rightarrow 0$  is split if (7)

$\cong$   ~~$0 \rightarrow B \rightarrow X_1 \rightarrow \dots \rightarrow X_k \rightarrow A \rightarrow 0$~~

$$0 \rightarrow B \rightarrow N_1 \rightarrow N_2 \rightarrow \dots \rightarrow N_k \rightarrow 0 \xrightarrow{\text{no } A} 0$$

$$\oplus 0 \rightarrow 0 \rightarrow \dots \rightarrow Q_j \rightarrow \dots \rightarrow Q_k \rightarrow A \rightarrow 0$$

$\sim \beta \uparrow$

Ex: For  $R = \mathbb{Q}[x]/x^2\text{-mod}$ , the generators of  $\text{Ext}^k(M_{(0)}, M_{(0)})$  is

$$0 \rightarrow M_{(0)} \xrightarrow{x} M_{(0)} \xrightarrow{x} \dots \xrightarrow{x} M_{(0)} \xrightarrow{x} M_{(0)} \xrightarrow{\overset{1}{x}} M_{(0)} \rightarrow 0$$

$\underset{R}{\parallel}$

What is

$$\begin{array}{ccccccccc} M_{(0)} & \rightarrow & R & \xrightarrow{ax} & R & \xrightarrow{bx} & R & \xrightarrow{ax} & R & \xrightarrow{1/x} & M_{(0)} & \rightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \text{circled } a_1^+ a_2^+ \dots a_k^+ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \dots & \rightarrow & R & \xrightarrow{x} & R & \xrightarrow{x} & R & \xrightarrow{x} & R & \xrightarrow{x} & R & \rightarrow & M_{(0)} \end{array}$$

Why care? Yoneda product  $\text{Hom}(B,C) \times \text{Hom}(A,B) \rightarrow \text{Hom}(A,C)$   
 $\text{Ext}^k(B,C) \times \text{Ext}^l(A,B) \rightarrow \text{Ext}^{k+l}(A,C)$

$\text{Ext}^*(\rightarrow)$  and here  $\text{Ext}^*$  is a graded algebra !!

$$0 \rightarrow C \rightarrow X_1 \rightarrow \dots \rightarrow X_k \xrightarrow{f} B \rightarrow 0 \rightsquigarrow 0 \rightarrow C \rightarrow X_1 \rightarrow \dots \rightarrow X_k \xrightarrow{g \circ f} Y_1 \rightarrow \dots \rightarrow Y_k \rightarrow A \rightarrow 0$$

$$0 \rightarrow B \xrightarrow{g} Y_1 \rightarrow \dots \rightarrow Y_k \rightarrow A \rightarrow 0$$

- obviously presaves equiv classes
- obviously assoc.
- not obviously additive (not terrible)



This defn doesn't work for  $\text{Ext}^0 := \text{Hom}$ , but for that map just

⑧

ie functoriality,

I.e.  $\text{Ext}^k(B, C) \times \text{Hom}(A, B) \rightarrow \text{Ext}^k(AC)$  is just functoriality of  $\text{Ext}^k(-, C)$ .

$\text{Ext}^*(A, A)$  is a graded algebra

Ex:  $R = \mathbb{C}[x]$   $M = M_{(0)}$ .  $\text{Ext}^i(M, M) = \begin{cases} \mathbb{C} & (i=0) \\ 0 & \text{etc.} \end{cases}$

$\text{Ext}^*(M, M) \cong \mathbb{C}[d]/d^2 = \Lambda^*(\mathbb{C})$

Why  $d^2=0$ ?  $(0 \rightarrow M_{(0)} \rightarrow M_{(0)} \rightarrow M_{(0)} \rightarrow 0)$ . (itself)

$= 0 \rightarrow M_{(0)} \xrightarrow{x} M_{(0)} \xrightarrow{x} M_{(0)}^{d^2/x^2} \rightarrow M_{(0)} \rightarrow 0$

$$\begin{array}{ccccccc} & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & & 1 & x & R & \xrightarrow{x} & R \rightarrow M_{(0)} \\ \oplus & & & & & & \\ 0 & \rightarrow & M_{(0)} & \rightarrow & M_{(0)} & \rightarrow & 0 \end{array}$$

equiv to split.

Ex:  $\Lambda = \Lambda^*(\mathbb{C}) = \mathbb{C}[d]/d^2$   $N = \mathbb{C} = \Lambda/d$   $\text{Ext}^i(N, N) = \begin{cases} \mathbb{C} & i=0 \\ 0 & \text{etc} \end{cases}$

Thm  $\text{Ext}^*(N, N) \cong R = \mathbb{C}[x]$

$x^k \leftrightarrow 0 \rightarrow N \xrightarrow{d} \Lambda \xrightarrow{d} \Lambda \xrightarrow{d} \Lambda \xrightarrow{d} \Lambda \xrightarrow{d} N \rightarrow 0$

is NOT split

(under natural iso to  $\text{Ext}^k$ )

Exercise:  $R = \mathbb{C}[x, y]$   $\Lambda = \Lambda^*(\mathbb{C}^2)$

then  $\text{Ext}_R^*(\mathbb{C}, \mathbb{C}) \cong \Lambda$  and  $\text{Ext}_\Lambda^*(\mathbb{C}, \mathbb{C}) \cong R$

Koszul duality.

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In real life there are better ways to compute the Ext algebra!

1)  $\mathbb{C}_X$  central sheaf on  $X$ .  $\text{Ext}^*(\mathbb{C}_X, \mathbb{C}_X) \cong H^*(X, \mathbb{C})$   
as rings.

2) cohomology of a dg algebra (later)

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HW Exercise:  $\mathbb{C}[x]/(x^2)$  compute  $\text{Ext}^*(M_{\mathbb{C}(x)}, M_{\mathbb{C}(x)})$

vs

$\mathbb{C}[x]/(x^k)$ ,  $k \geq 2$ , compute  $\text{Ext}^*(M_{\mathbb{C}(x)}, M_{\mathbb{C}(x)})$

quadratic is special!