Yoneda Ext | ? I gotta ext. | Why ext is called ext.

First: more on pushouts.

Given:
$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f"'] & P \arrow[d, "\underset{\sim}{f}"] \\ B \arrow[r, "\underset{\sim}{g}"'] & X \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$
and
$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f'"'] & P \arrow[d] \\ B \arrow[r] & X' \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$
, when is
$$\begin{tikzcd} & X \arrow[d, "\wr"] \\ B \arrow[ur] \arrow[r] & X' \end{tikzcd}$$
?

Here's a situation where it happens: Spose $P \xrightarrow{h} B$. Let $f' = f + hg$.

$$X = \mathrm{Coeq}\,(M \rightrightarrows B \oplus P) = B\oplus P \Big/ \underbrace{\langle (f(m), -g(m)) \rangle}_{} = I$$

$$X' = B\oplus P \Big/ \langle (f(m) + hg(m), -g(m)) \rangle = I'$$

$B \oplus P$ has autom. $\begin{pmatrix} 1_B & -h \\ 0 & 1_P \end{pmatrix}$ which preserves B and sends $\begin{pmatrix} b \\ p \end{pmatrix} \mapsto \begin{pmatrix} b - hp \\ p \end{pmatrix}$

so sends I to I'. Induces $X \xrightarrow{\sim} X'$.

Said another way, X would also serve as the pushout of diagram 2,

with
$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "f+hg"'] & P \arrow[d, "\underset{\sim}{f} + \underset{\sim}{g}h"] \\ B \arrow[r, "\underset{\sim}{g}"'] & X \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$
(sure looks like a homotopy to me!)

$$\text{Pushouts } \begin{tikzcd}[column sep=small, row sep=small] M \arrow[r, "g"] \arrow[d] & P \\ B & \end{tikzcd} \Big/ \cong \text{rel } B \longleftarrow \mathrm{Hom}(M,B) \Big/ \mathrm{Hom}(P,B)\circ g .$$

This is isom, one moment please.

Def: An extension of A by B is $0\to B \xrightarrow{f} X \xrightarrow{g} A \to 0$

equivalent if

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "g"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "f'"] & X' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

(note: it's an isom, but let's view that as an accident...)

split if equiv to $0\to B\to B\oplus A\to A\to 0$.

$\mathrm{Exten}^1(A,B) := \text{Extensions}/\cong$.

Thm (Yoneda Baer): 1) $\mathrm{Exten}^1(A,B)$ is an abelian group (as below)!

2) $\mathrm{Exten}^1(A,B) \cong \mathrm{Ext}^1(A,B)$ as ab gps functorially in A and B.

First we discuss the bijection.

Given $0\to B\to X\to A\to 0$ apply $\mathrm{Hom}(A,-)$ to get les

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(A,X) \arrow[r] & \mathrm{Hom}(A,A) \arrow[r, "\delta_A"] & \mathrm{Ext}^1(A,B) \arrow[r] & \cdots \\
 & & & 1_A \arrow[r, mapsto] & \delta_A(1_A) &
\end{tikzcd}$$

Functoriality of les says indep of equiv class

OR apply $\mathrm{Hom}(-,B)$ to get

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[r, "\delta_B"] & \mathrm{Ext}^1(A,B) \arrow[r] & \cdots \\
 & & & 1_B \arrow[r, mapsto] & \delta_B(1_B) &
\end{tikzcd}$$

Under future identification of two versions of $\mathrm{Ext}^1(A,B)$, $\delta_A(1_A)$ matches $\delta_B(1_B)$.

Another view: $(P^\bullet) \twoheadrightarrow A$ then comp. lemma gives

$$\begin{tikzcd}
\cdots \arrow[r] & P_{-1} \arrow[r] \arrow[d, "\chi"] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

It is $\chi \in \mathrm{Hom}(P_{-1},B)$ which gives a class in $\mathrm{Hom}(P^\bullet,B)$ descending to $\delta_B(1_B)$ in $h^1(\mathrm{Hom}(P^\bullet,B)) = \mathrm{Ext}^1(A,B)$

Another view! $0 \to M \xrightarrow{g} P \to A \to 0$ w/ P projective.

For any [illegible] right exact F we have

$$\cdots 0 \to L^2FA \to L^1FM \to 0 \to L^1FA \to FM \to FP \to FA \to 0$$

dimensional reduction.

So $\mathrm{Ext}^1(A,B) \cong \mathrm{Hom}(M,B) / \mathrm{Hom}(P,B)\circ g$

Given $0 \to B \to X \to A \to 0$ get

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\rho"'] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

and $\rho \in \mathrm{Hom}(M,B)$ descends to $\delta_B(1_B)$ in $\mathrm{Ext}^1(A,B)$.

☆ $\boxed{\text{GO TO } (3')}$ ☆

Conversely, given $f \in \mathrm{Hom}(M,B)$ can construct pushout

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near end] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

b/c pushout preserves cokernel.

this yields an extension. As noted, gives a map $\mathrm{Ext}^1(A,B) = \mathrm{Hom}(M,B) / \mathrm{Hom}(P,B)\circ g$ to $\mathrm{Exten}^1(A,B)$

Claim: These maps are inverse, i.e. if

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"'] & P \arrow[r] \arrow[d, "j"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "q"'] & X \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

then $X \cong$ pushout (rel B)

Pf: Have map $B \oplus P \xrightarrow{q \oplus j} X$ and kernel contains $\langle (g(m), -f(m)) \rangle$

I'll use elements to show isom.

- $q \overset{*}{\oplus} j$ surjective: if $x \in X$, $x \mapsto a \in A$ then $\exists p \in P$, $p \mapsto a$

  $x - j(p) \mapsto 0$ so $\exists b \in B$, $b \mapsto x - j(p)$.

- if $(b,p) \mapsto 0 \in X$ then $q(b) = j(p)$. $\Rightarrow$ $j(p) \mapsto 0 \in A$

  $\Rightarrow$ $p \mapsto 0 \in A$

  $\Rightarrow \exists\, m \in M$, $m \mapsto p$. B/c $B \to X$ is injective $\begin{array}{c} m \\ \downarrow \\ b \end{array}$. ∎

We have the bijection. How is Exten' functorial ??

Given $A \xrightarrow{x} A'$ want $\mathrm{Exten}^1(A',B) \to \mathrm{Exten}^1(A,B)$

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[r] \arrow[d, "x"] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

use pullback square.

Functoriality of les says this commutes with map to $\mathrm{Ext}^1(-,B)$

Given $B \xrightarrow{\Delta} B'$ want $\mathrm{Exten}^1(A,B) \to \mathrm{Exten}^1(A,B')$

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B' \arrow[r, dashed] & X' \arrow[r, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & A \arrow[r] & 0
\end{tikzcd}$$

Use pushout.

How is $\mathrm{Exten}^1(A,B)$ an abelian group ???

<u>Baer Sum</u>: Given $0 \to B \xrightarrow{\varphi} X \xrightarrow{q} A \to 0$ $\quad 0 \to B \xrightarrow{\varphi'} X' \xrightarrow{q'} A \to 0$

Consider

$$\begin{tikzcd}
X \times_A X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "q'"] \\
X \arrow[r, "q"'] & A
\end{tikzcd}$$

$X \times_A X' = \mathrm{Ker}\left(X \oplus X' \xrightarrow[q - q']{} A\right)$

Now B actually has lots of maps to $X \times_A X'$, two in particular.

$B \to X \times_A X'$ and $B \to X \times_A X'$. Let's make them equal.

$b \mapsto (\varphi(b), 0)$ $\qquad$ $b \mapsto (0, \varphi'(b))$

Let $Y = X \times_A X' \big/ \langle (g(b), -g'(b)) \rangle$ i.e. $(g(b),0) = (0,g'(b))$ in quotient.

Then $0 \to B \to Y \to A \to 0$ is exact (exercise).

either $b \mapsto (g(b),0)$ or $(0,g'(b))$

Y is the Baer Sum of X and X'!

Comparison w/ Ext^1: Fix $0 \to M \to P \to A \to 0$ and consider

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"'] & P \arrow[r] \arrow[d, "j"'] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f'"'] & P \arrow[r] \arrow[d, "j'"'] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Then $P \xrightarrow[j \oplus j']{} X \oplus X'$ has image in $X \times_A X'$, so get map $P \to Y$

Exercise:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f+f'"'] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

$f+f'$ is induced map.

Identity is the split extension!

1) For Ext^1, $\delta = 0$ for split ses (Why?) (i.e. splits $0 \to B \to B \to 0 \oplus 0 \to A \to A \to 0$)

2) For Baer, $X \times_A (B \oplus A) \cong X \oplus B$

$$Y = X \oplus B \big/ (g(b),0) = (0,b) \cong X.$$

$$\boxed{☆ \text{ Go to } ⑤' ☆}$$

3'

Ex: $\mathbb{C}[x]$-mod.

$$\begin{tikzcd}
0 \arrow[r] & M_{(\lambda)} \arrow[r, "e_1"] & M_{\begin{pmatrix}\lambda & c\\ 0 & \lambda\end{pmatrix}} \arrow[r, "/\langle e_1\rangle"] & M_{(\lambda)} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C}[x] \arrow[u, "c"] \arrow[r, "x-\lambda"] & \mathbb{C}[x] \arrow[u, "e_2"] \arrow[r] & M_{(\lambda)} \arrow[u] \arrow[r] & 0
\end{tikzcd}$$

[illegible]

$$1 \mapsto (x-\lambda)e_2 = ce_1$$

Ex: Baer sum.

5'

$$M_{\begin{pmatrix}\lambda & c\\ & \lambda\end{pmatrix}} \underset{M_{(\lambda)}}{\times} M_{\begin{pmatrix}\lambda & c'\\ & \lambda\end{pmatrix}}$$

$$= \{ (ae_2 + ke_1, ae_2 + k'e_1) \}$$

3D

$$\begin{array}{ccc} \lambda & & c \\ & \lambda & c' \\ & & \lambda \\ (e_1,0) & (0,e_1) & (e_2,e_2) \end{array}$$

diagonal image of $M_{(\lambda)}$ is $(ke_1, -ke_1)$

so $Y =$ Span $\{ (e_2,e_2), (e_1,0) \}$ in quotient

$$(x-\lambda)(e_1,0) = 0$$

$$(x-\lambda)(e_2,e_2) = (ce_1, c'e_1) = ((c+c')e_1, 0).$$

Great... what about $\mathrm{Ext}^2(A,B)$??

Def: $\mathrm{Exten}^2(A,B) = \{ \underset{\text{exact}}{0 \to B \to X_1 \to X_2 \to A \to 0} \} / \text{equiv}$

where: primitively equiv if

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X_1' \arrow[r] & X_2' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

(not necc isoms!!)

This isn't equiv reln, so take equiv reln generated by this.

Similarly, $\mathrm{Exten}^k(A,B) = \{ 0 \to B \to X_1 \to \cdots \to X_k \to A \to 0 \} / \text{equiv}$

Thm (Y): $\mathrm{Exten}^k(A,B) \cong \mathrm{Ext}^k(A,B)$ as abelian grps, functorially.

How? Choose

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

dimensional reduction: $\mathrm{Ext}^2(A,B) \cong \mathrm{Hom}(M,B) / \mathrm{Hom}(P_1,B)$.

exists by comparison lemma (the map $M \to B$ induced via kernels)

Conversely, given $f \in \mathrm{Hom}(M,B)$ get

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near end] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Check: for bijection, exists

Pf: Ginormous pushback exercise...

<u>Note:</u> $0 \to B \to X_1 \to \cdots \to X_k \to A \to 0$ is <u>split</u> if

$$\begin{tikzcd} & & & & & & & \text{no } A \arrow[d] & \\ \cong & 0 \arrow[r] & B \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & \cdots \arrow[r] & N_\ell \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

$\oplus$

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & Q_j \arrow[r] & \cdots \arrow[r] & Q_k \arrow[r] & A \arrow[r] & 0 \\ \sim B \arrow[ru] & & & & & & & \end{tikzcd}$$

<u>Ex:</u> For $R = \mathbb{C}[x]/x^2\text{-mod}$, the generator of $\mathrm{Exten}^k(M_{(0)}, M_{(0)})$ is

$$\begin{tikzcd} 0 \arrow[r] & M_{(0)} \arrow[r, "x"] & M_{(0,1)} \arrow[r, "x"] \arrow[d, phantom, "\parallel"] & \cdots \arrow[r, "x"] & M_{(0,1)} \arrow[r, "x"] & M_{(0,1)} \arrow[r, "x"] & M_{(0,1)} \arrow[r, "/(x)"] \arrow[d, phantom, "\parallel"] & M_{(0)} \arrow[r] & 0 \\ & & R & & & & R & & \end{tikzcd}$$

(What is

$$\begin{tikzcd} M_{(0)} \arrow[r] & R \arrow[r, "a_k x"] & R \arrow[r, "a_2 x"] & R \arrow[r, "a_1 x"] & R \arrow[r, "/(x)"] & M_{(0)} \arrow[r] & 0 \\ \cdots \arrow[r] & R \arrow[r, "x"] \arrow[lu, "a_1^{-1}a_2^{-1}\cdots a_k^{-1}"] & R \arrow[r, "x"] \arrow[lu] & R \arrow[r, "x"] \arrow[lu, "a_1^{-1}a_2^{-1}"'] & R \arrow[r, "x"] \arrow[lu, "a_1^{-1}"'] & R \arrow[r] \arrow[lu, "1"'] & M_{(0)} \arrow[lu, equal] \end{tikzcd}$$

)

Why care? <u>Yoneda product</u>

Ex: $\mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$

$$\begin{tikzcd} \mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B) \arrow[r] \arrow[d, squiggly] & \mathrm{Hom}(A,C) \arrow[d, squiggly] \\ \mathrm{Exten}^k(B,C) \times \mathrm{Exten}^\ell(A,B) \arrow[r] & \mathrm{Exten}^{k+\ell}(A,C) \end{tikzcd}$$

$\mathrm{Exten}^*(-,-)$ and hence Ext^* is a graded algebra !!

$0 \to C \to X_1 \to \cdots \to X_k \xrightarrow{p} B \to 0$ $\rightsquigarrow$ $0 \to C \to X_1 \to \cdots \to X_k \xrightarrow{q \circ p} Y_1 \to \cdots \to Y_\ell \to A \to 0$

$0 \to B \xrightarrow{q} Y_1 \to \cdots \to Y_\ell \to A \to 0$

- obviously preserves equiv classes
- obviously assoc.
- not obviously additive (not terrible)

This defn doesn't work for $\mathrm{Ext}_{Yon}^0 := \mathrm{Hom}$, but for that map just use functoriality,

I.e. $\mathrm{Ext}_{Yon}^k(B,C) \times \mathrm{Hom}(A,B) \longrightarrow \mathrm{Ext}_{Yon}^k(A,C)$ is just functoriality of $\mathrm{Ext}_{Yon}^k(-,C)$.

$\mathrm{Ext}^*(A,A)$ is a graded algebra

<u>Ex:</u> $R = \mathbb{C}[x]$ $M = M_{(0)}$. $\mathrm{Ext}^i(M,M) = \begin{cases} \mathbb{C} & i = 0,1 \\ 0 & \text{else.} \end{cases}$

so $\mathrm{Ext}^*(M,M) \cong \mathbb{C}[d]/d^2 = \Lambda^*(\mathbb{C})$

Why $d^2 = 0$? $\left(0 \to M_{(0)} \to M_{(0,1)} \to M_{(0)} \to 0\right) \cdot \left(\text{ itself}\right)$

$$\begin{tikzcd}
= \; 0 \arrow[r] & M_{(0)} \arrow[r, "x"] & M_{(0,1)} \arrow[r, "x"] & \overset{\mathbb{C}[x]/x^2}{\overset{\parallel}{M''_{(0,1)}}} \arrow[r] & M_{(0)} \arrow[r] & 0 \\
 & & R \arrow[u, "1"] \arrow[r, "x"] & R \arrow[u, "1"] \arrow[r] & M_{(0)} \arrow[u] & \\
\oplus & & \oplus & & & \\
0 \arrow[r] & M_{(0)} \arrow[uuu, "1"] \arrow[r] & M_{(0)} \arrow[uuu, bend left, "x"] \arrow[r] & 0 & &
\end{tikzcd}$$

equiv to split.

<u>Ex:</u> $\Lambda = \Lambda^*(\mathbb{C}) = \mathbb{C}[d]/d^2$ $N = \mathbb{C} = \Lambda/(d)$ $\mathrm{Ext}^i(N,N) = \begin{cases} \mathbb{C} & i \geq 0 \\ 0 & \text{else} \end{cases}$

<u>Thm</u> $\mathrm{Ext}^*(N,N) \cong R = \mathbb{C}[x]$

$x^k \Longleftrightarrow 0 \to N \xrightarrow{d} \Lambda \xrightarrow{d} \Lambda \xrightarrow{d} \Lambda \xrightarrow{d} \Lambda \xrightarrow{(k)} N \to 0$ is NOT split

(induces natural iso to Ext^k)

Exercise: $R = \mathbb{C}[x,y]$ $\quad \Lambda = \Lambda^*(\mathbb{C}^2)$

then $\mathrm{Ext}^*_R(\mathbb{C},\mathbb{C}) \cong \Lambda$ and $\mathrm{Ext}^*_\Lambda(\mathbb{C},\mathbb{C}) \cong R$

Koszul duality.

In real life there are better ways to compute the Ext algebra!

1) $\underline{\mathbb{C}}_X$ constant sheaf on X. $\mathrm{Ext}^*(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X) \cong H^*(X;\mathbb{C})$
as rings.

2) cohomology of a dg algebra (later)

HW Exer: $\mathbb{C}[x]/(x^2)$ compute $\mathrm{Ext}^*(M_{(0)}, M_{(0)})$

vs

$\mathbb{C}[x]/(x^k)$, $k > 2$, compute $\mathrm{Ext}^*(M_{(0)}, M_{(0)})$

quadratic is special!