QUACKS Exercises

Lecture 1

1. (Hopf algebras vs. Hopf superalgebras) Let H be an algebra over a field $\Bbbk$. One of the axioms of a Hopf algebra is that the map $\Delta\colon H \to H \otimes_{\Bbbk} H$ is an algebra homomorphism. This raises the question: what is the algebra structure on $H \otimes H$?

a) For a normal Hopf algebra, the algebra structure is

$$(a \otimes b) \cdot (c \otimes d) := (ac) \otimes (bd). \tag{1}$$

Let $H = \Bbbk[d]/(d^p)$ where p is a prime. Prove that $\Delta(d) = d \otimes 1 + 1 \otimes d$ extends to an algebra homomorphism if and only if $\Bbbk$ has characteristic p.

b) For a super Hopf algebra, H is graded, and $|a|$ denotes the degree of a homogeneous element $a \in H$. The algebra structure on $H \otimes H$ is

$$(a \otimes b) \cdot (c \otimes d) := (-1)^{|b||c|}(ac) \otimes (bd). \tag{2}$$

Let $H = \Bbbk[d]/(d^2)$ with $\deg d = 1$, and let $\Delta(d) = d \otimes 1 + 1 \otimes d$. Prove that Δ extends to an algebra homomorphism, for any base field $\Bbbk$.

c) For a quantum Hopf algebra over $\Bbbk$, H is graded, and $|a|$ denotes the degree of a homogeneous element $a \in H$. The algebra structure on $H \otimes H$ is

$$(a \otimes b) \cdot (c \otimes d) := q^{|b||c|}(ac) \otimes (bd), \tag{3}$$

for some fixed element $q \in \Bbbk$. Let $H = \Bbbk[d]/(d^p)$ where p is a prime, and let $\deg d = 1$. What property of q is equivalent to asking that $\Delta(d) = d \otimes 1 + 1 \otimes d$ extends to an algebra homomorphism?

Remark 0.1. The fact that one must work in the category of super vector spaces in order for $\Bbbk[d]/(d^2)$ to be a Hopf algebra explains all the sign rules which appear in homological algebra.

2. Here are some general facts to prove. Let H be a finite-dimensional Hopf algebra, and M be an H-module.

a) Prove that $M[1][-1] \cong M$. (The definition of the shift is found on slide 16.)

b) Recall that an *integral* $\Lambda \in H$ is an element whose ideal (Λ) is one-dimensional, and (Λ) is isomorphic to the trivial representation $\Bbbk$ as an H-module, i.e. $h\Lambda = \epsilon(h)\Lambda$. For any H-module M, prove that the map

$$i_M\colon M \to H \otimes M, \qquad m \mapsto \Lambda \otimes m, \tag{4}$$

is an H-module inclusion.

c) In fact, $H \otimes M$ is always a free module: prove this in the special case when $H = \Bbbk[d]/(d^2)$. So $i_M\colon M \to H \otimes M$ is a functorial way to embed a module into a free module.

d) Prove that a short exact sequence of H-modules gives rise to a distinguished triangle in the stable category.

3. Let $H = \Bbbk[d]/(d^2)$ be the super Hopf algebra. We interpret graded modules over H as complexes of vector spaces over $\Bbbk$.

a) A complex of vector spaces over $\Bbbk$ is called *free* if it is a direct sum of complexes of the form $(0 \to \Bbbk \xrightarrow{\sim} \Bbbk \to 0)$ with various homological shifts. Why is this a good name?

b) The previous exercise gives a functorial map i_M from any complex M to a free complex. Describe this explicitly.

c) Prove that a chain map between complexes is nulhomotopic if and only if it factors through a free complex (and more precisely, if it factors through the functorial map).

d) Look up the definition of the cone of a chain map f. Match this definition with the pushout C_f defined in lecture (slide 17).

e) On slide 20, there is a theorem which (in this context) describes the space of chain maps modulo homotopy. Match this with the familiar definition of chain maps modulo homotopy.

4. Now let $H = \Bbbk[d]/(d^p)$ in characteristic p, with $\deg d = 1$.

a) Show that a morphism $f\colon M \to N$ of graded H-modules is nulhomotopic if and only if there is a linear map $h\colon M \to N$ of degree $-(p-1)$ such that

$$f = \sum_{i=0}^{p-1} d^i h d^{p-1-i}. \tag{5}$$

b) Reinterpret the theorem on slide 20: the space of morphisms $M \to N$ in the stable category is a subquotient of the space of all $\Bbbk$-linear maps, given by taking the kernel of __ modulo the image of __.

c) Classify the indecomposable graded H-modules (hint: Jordan normal form). For each pair of indecomposable H-modules, compute the space of morphisms between them in the category of graded H-modules, and also in the stable category. There are a lot of cases here (don't forget grading shifts!), but you don't have to do them all, just do enough that you get the point. Try to draw your morphisms as chain maps between p-complexes.

Advanced exercise 1. (Don't expect any help!) What is the integral inside the small quantum group $u(\mathfrak{sl}_2)$ (either in finite characteristic, or at a root of unity)?

Advanced exercise 2. (Don't expect any help!) Let $H = \Bbbk\langle d_1, d_2\rangle/(d_1^2, d_2^2, d_1 d_2 + d_2 d_1)$ be the exterior algebra on two generators, a finite-dimensional bigraded super Hopf algebra. We think of bigraded H-modules as bicomplexes. Describe the bicomplexes associated to the trivial module and the free module. Find all shifts of the trivial module. Classify the indecomposable modules, and describe their shifts. Give several homological functors from the stable category to vector spaces.

Advanced exercise 3. (Don't expect any help!) Prove axioms (TR0)-(TR4) of a triangulated category for the stable module category.

Advanced exercise 4. (Don't expect any help!) Categorify other interesting rings using finite-dimensional Hopf algebras.