Categorification at a prime root of 1

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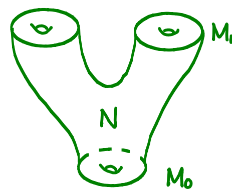
Lecture 1. Background, categorifying $\mathcal{O}_p$.
Lecture 2 Categorifying $U_q(\mathfrak{sl}_2)$ and Weyl modules.
Lecture 3. Towards tensor products

🚩 Background: categorification at roots of unity

Topological Quantum Field Theories

Atiyah, Segal etc.



$$\xRightarrow{Z} \qquad Z(M_0) \xrightarrow{Z(N)} Z(M_1)$$

subject to coherence axioms.

Examples

- In dimension 3. Chern-Simons-Witten, Jones
  Reshetikhin-Turaev, Turaev-Viro
  Kuperberg, Henning, Kauffman etc.

- In dimension 4. Donaldson-Floer, Seiberg-Witten etc.

Crane-Frenkel Conjecture

$$
\begin{tikzcd}
\text{Donaldson-Floer} \arrow[d, Rightarrow] & \boxed{\text{A 4D TQFT ?}} \arrow[l, Rightarrow] \\
\begin{array}{l}\text{Kuperberg: } U_q^{+}(sl_2)\text{-invariant} \\ \text{of 3-manifolds}\end{array} & \begin{array}{c}\text{WRT-3D TQFT} \\ \text{associated with } U_q(sl_2)\end{array} \arrow[l, Rightarrow] \arrow[u, Rightarrow, "\text{Categorification}"']
\end{tikzcd}
$$

- q: a primitive n-th root of unity

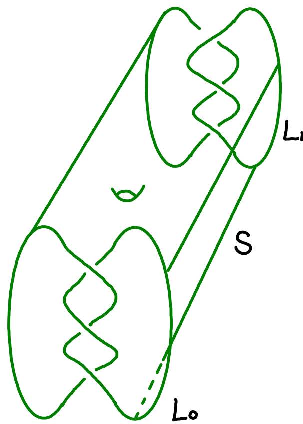
Some Evidence:

- Khovanov homology and generalizations.
  - A functorial link invariant at a generic q value.

- Heegaard-Floer homology of Ozsvath-Szabo.
  - A combinatorial construction of Seiberg-Witten theory.

  - Categorical invariants for links and 3-manifolds.

Dividend: functoriality



$$\mathrm{Kh}(L_0)\ (\mathrm{HFK}(L_0)) \xrightarrow{\mathrm{Kh}(S)\ (\mathrm{HFK}(S))} \mathrm{Kh}(L_1)\ (\mathrm{HFK}(L_1))$$

$$\overset{\chi}{\Longrightarrow}$$

$$\mathrm{Jones}(L_0)\ (\mathrm{Alexander}(L_0)) \dashrightarrow \mathrm{Jones}(L_1)\ (\mathrm{Alexander}(L_1))$$

Digression: Homological Algebra

Basic features of homological algebra (over a field)

(0) $\mathrm{Kom}(\Bbbk)$: chain complexes $(K^\bullet, d)$: $d^2=0 \implies H^\bullet(K^\bullet)$

(1) $K^\bullet, L^\bullet \in \mathrm{Kom}(\Bbbk) \implies K^\bullet \oplus L^\bullet \in \mathrm{Kom}(\Bbbk)$

$d(k,\ell) := (d(k), d(\ell))$

(2) $K^\bullet, L^\bullet \in \mathrm{Kom}(\Bbbk) \implies K^\bullet \otimes L^\bullet \in \mathrm{Kom}(\Bbbk)$

$d(k\otimes\ell) := d(k)\otimes\ell + (-1)^{|k|} k\otimes d(\ell)$

(3) $K^\bullet, L^\bullet \in \mathrm{Kom}(\Bbbk) \implies \mathrm{HOM}^\bullet(K^\bullet, L^\bullet) \in \mathrm{Kom}(\Bbbk)$

$d(f)(k) := d(f(k)) - (-1)^{|f|} f(d(k))$

(4) Triangulated structure: $[1]$, cones, s.e.s. $\rightsquigarrow$ d.t.

(TR1 – TR4) etc.

Why so useful in categorification?

- A biased reason:

$$\begin{array}{lccc}
 & \mathrm{Com}(\Bbbk) := \mathrm{Kom}(\Bbbk)/\sim & \xrightarrow{\chi} & \mathbb{Z} \\
\text{Variant} & g\mathrm{Com}(\Bbbk) & \longrightarrow & \mathbb{Z}[q,q^{-1}] \\
 & K^\bullet & \longmapsto & \chi(K^\bullet) \\
 & \oplus & \longmapsto & + \\
 & \otimes & \longmapsto & \times \\
 & [1] & \longmapsto & -1 \\
 & \{1\} & \longmapsto & q
\end{array}$$

Observation: features (1)-(3) are rather reminiscent of representation theory of Hopf algebras.

Def A $\Bbbk$-algebra H is called a Hopf algebra if there is are algebra homomorphisms $\Delta: H \longrightarrow H\otimes H$ (comultiplication) $\epsilon: H \longrightarrow \Bbbk$ (counit), $S: H \longrightarrow H^{op}$ s.t

(1)
$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & H^{\otimes 2} \arrow[d, "\Delta\otimes \mathrm{Id}"] \\
H^{\otimes 2} \arrow[r, "\mathrm{Id}\otimes\Delta"'] & H^{\otimes 3}
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "="] & H^{\otimes 2} \arrow[d, "\epsilon\otimes \mathrm{Id}"] \\
H^{\otimes 2} \arrow[r, "\mathrm{Id}\otimes\epsilon"'] & H
\end{tikzcd}$$

(3) $\forall h \in H$, write $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$,

$$\sum h_{(1)} S(h_{(2)}) = \epsilon(h) = \sum h_{(1)} S(h_{(2)}).$$

(H, Δ, ϵ, S) : Hopf algebra.

(1). $K, L \in H\text{-mod} \Longrightarrow K \oplus L \in H\text{-mod}$

$$h \cdot (k, \ell) = (h \cdot k, h \cdot \ell)$$

(2). $K, L \in H\text{-mod} \Longrightarrow K \otimes L \in H\text{-mod}$

$$h \cdot (k \otimes \ell) = \sum h_{(1)} k \otimes h_{(2)} \ell$$

(3). $K, L \in H\text{-mod} \Longrightarrow \mathrm{HOM}(K, L) \in H\text{-mod}$

$$(h \cdot f)(k) = \sum h_{(2)} f(S^{-1}(h_{(1)}) k)$$

Examples

(1). $H = \Bbbk G$, group algebra

$\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g^{-1}$.

(2). G: compact Lie group. $H^*(G, \Bbbk)$ is a Hopf superalgebra.

(e.g. $G = U(n)$, $H^*(U(n), \Bbbk) \cong \Lambda^*(d_1, \cdots, d_n)$, $\deg(d_i) = 2i-1$.

$\Delta(d_i) = d_i \otimes 1 + 1 \otimes d_i$, $\epsilon(d_i) = 0$, $S(d_i) = -d_i$.)

(3). $\mathfrak{g}$: Lie algebra, $U(\mathfrak{g}) := T(\mathfrak{g}) / (x \otimes y - y \otimes x - [x, y])$ is a Hopf algebra: $\forall x \in \mathfrak{g}$, $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$, $S(x) = -x$.

Slogan: Homological algebra

" =" Representation theory of the graded Hopf superalgebra $\Bbbk[d]/(d^2) \cong H^*(S^1, \Bbbk)$

Question: Are there other features of homological algebra present for H-modules?

Cohomology for H-mod? Triangulated structure?

Cohomology for chain complexes

Any chain complex $(K^\bullet, d)$ decomposes:

$$(K^\bullet, d) \cong (0 \longrightarrow \mathbb{k} \longrightarrow 0)^{\oplus r} \oplus \underbrace{(0 \longrightarrow \mathbb{k} \xrightarrow{\cdot 1} \mathbb{k} \longrightarrow 0)^{\oplus s}}_{H^\bullet(-) \text{ kills these}}$$

$(0 \longrightarrow \mathbb{k} \xrightarrow{\cdot 1} \mathbb{k} \longrightarrow 0)$:

- Projective graded $\mathbb{k}[d]/(d^2)$-modules
- They are also injective!

Question: (1) When are projective H-modules also injective?

(2). If $\mathrm{Proj}(H) = \mathrm{Inj}(H)$, how do we "kill" them ?

Thm (Larson-Sweedler) H: Hopf algebra/$\mathbb{k}$. Then H is Frobenius iff H is finite-dim'l.

In particular, for finite-dim'l Hopf algebras, projective H-modules coincide with injective H-modules.

• The stable category and hopfological algebra

H: finite dim'l Hopf algebra.

Def. The stable category $H\text{-}\underline{\mathrm{mod}}$ has the same objects as $H\text{-}\mathrm{mod}$, while for any $K, L \in H\text{-}\underline{\mathrm{mod}}$,

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K,L) := \frac{\mathrm{Hom}_H(K,L)}{\left\{ \begin{tikzcd}[row sep=small, column sep=small] K \arrow[rr] \arrow[dr] & & L \\ & P \arrow[ur] & \end{tikzcd} \,\middle|\, P : \mathrm{proj} \right\}} .$$

Thm (Heller) H: Frobenius $\Longrightarrow$ H-$\underline{\text{mod}}$ is triangulated.

Proof sketch:

- Shift functors. $M \in H$-mod, choose an injective embedding and a projective covering
$$0 \longrightarrow M \xrightarrow{\alpha} I_M \,, \qquad P_M \xrightarrow{\beta} M \longrightarrow 0,$$
and define $M[1] := \operatorname{coker} \alpha$, $M[-1] := \ker \beta$.

- Distinguished triangles: if $f: K \longrightarrow L$ is a map of H-modules

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
K \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\#"] & L \arrow[d] \\
I_K \arrow[r] \arrow[d] & C_f \arrow[d] \\
K[1] \arrow[r, equal] \arrow[d] & K[1] \arrow[d] \\
0 & 0
\end{tikzcd}
\qquad \xRightarrow{\text{declare}} \qquad
K \xrightarrow{f} L \longrightarrow C_f \longrightarrow K[1]
$$

a standard d.t.

Question: How do we compute morphism spaces explicitly?

Def. An element $\Lambda \in H$ is called a (left) integral if $\forall h \in H$,

$$h \cdot \Lambda = \epsilon(h) \Lambda .$$

Thm. (Larson-Sweedler) H: finite dim'l $\Longrightarrow$

$$\dim \{\Lambda \mid \Lambda : \text{left integral}\} = 1 .$$

Examples

(1). $H = \mathbb{k}G$, G: finite group. $\Lambda = \sum_{g \in G} g$.

(2). $H = \mathbb{k}[d]/(d^2)$ (graded Hopf superalgebra) $\Lambda = d$.

(3). $H = \mathbb{k}[\partial]/(\partial^p)$ (graded Hopf algebra if $\mathrm{char}\,\mathbb{k} = p > 0$)

Thm. (Q) $\forall K, L \in H\text{-mod}$

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K,L) := \frac{\mathrm{HOM}(K,L)^H}{\Lambda \cdot \mathrm{HOM}(K,L)}.$$

Proof: reduces to the following lemmas.

Lem 1. $\mathrm{HOM}(K,L)^H = \mathrm{Hom}_H(K,L)$.

Pf. "$\supseteq$": $f \in \mathrm{Hom}_H(K,L) \Longrightarrow$

$$(h \cdot f)(-) = h_{(2)} f(S^{-1}(h_{(1)}) \cdot (-)) = h_{(2)} S^{-1}(h_{(1)}) f(-) = \epsilon(h) f(-).$$

"$\subseteq$" $f \in \mathrm{HOM}(K,L)^H \Longrightarrow$

$$f(h \cdot (-)) = \epsilon(h_{(2)}) f(h_{(1)} \cdot (-)) = (h_{(2)} \cdot f)(h_{(1)}(-)) = h_{(3)} f(S^{-1}(h_{(2)}) h_{(1)}(-))$$

$$= \epsilon(h_{(1)}) h_{(2)} f(-) = h \cdot f(-)$$

$\square$

Lem 2. $f \in \mathrm{Hom}_H(K,L)$ factors through a projective H-module iff f factors as

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "\mathrm{Id}\otimes\Lambda"'] & & L \\ & K\otimes H \arrow[ur, "\tilde{f}"'] & \end{tikzcd}.$$

Pf: Suffices to show for L projective, or even $L = H$.

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "\mathrm{Id}_K\otimes\Lambda"'] & H \arrow[d, "\mathrm{Id}_H\otimes\Lambda"] \\ K\otimes H \arrow[r, "f\otimes \mathrm{Id}"] & H\otimes H \arrow[u, dashed, bend left, "g"] \end{tikzcd}$$

H injective $\Longrightarrow \exists$ splitting $g \Longrightarrow f = g\circ(f\otimes \mathrm{Id})\circ(\mathrm{Id}_K\otimes\Lambda)$. $\square$

Lem 3. $f \in \mathrm{Hom}_H(K, L)$ factors through as

$$\begin{tikzcd} K \arrow[rr, "f"] \arrow[dr, "\mathrm{Id}\otimes\Lambda"'] & & L \\ & K\otimes H \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

iff $f = \Lambda \cdot \varphi$, where $\varphi \in \mathrm{HOM}(K, L)$.

Pf: "$\Longrightarrow$" $\tilde{f}$ given. define $\varphi = \tilde{f}|_{K\otimes \Bbbk}$ Then

$$(\Lambda\cdot\varphi)(k) = \Lambda_{(2)}\varphi(S^{-1}(\Lambda_{(1)})k) = \Lambda_{(2)}\tilde{f}(S^{-1}(\Lambda_{(1)})k\otimes 1) = \tilde{f}(\Lambda_{(2)}(S^{-1}(\Lambda_{(1)})k\otimes 1)$$

$$= \tilde{f}(\Lambda_{(2)}S^{-1}(\Lambda_{(1)})k\otimes\Lambda_{(3)}) = \tilde{f}(\epsilon(\Lambda_{(1)})k\otimes\Lambda_{(2)}) = \tilde{f}(k\otimes\Lambda)$$

"$\Longleftarrow$" Exercise. $\square$

Examples (ct'd)

(1) $\Bbbk G$: semisimple $\iff$ $\underline{\Bbbk}$ is projective (injective)

$\left(\because \ \mathrm{Hom}_H(M,-) \cong \mathrm{Hom}_H(\underline{\Bbbk}\otimes M,-) \cong \mathrm{Hom}_H(\underline{\Bbbk}, \mathrm{HOM}(M,-))\right)$

$\iff \mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(\underline{\Bbbk}, \underline{\Bbbk}) = 0$

But

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(\underline{\Bbbk}, \underline{\Bbbk}) = \frac{\mathrm{HOM}(\Bbbk,\Bbbk)^H}{\Lambda\cdot \mathrm{HOM}(\Bbbk,\Bbbk)} = \frac{\Bbbk}{|G|\cdot \Bbbk}$$

Thus $\Bbbk G$ semisimple $\iff$ $|G| \in \Bbbk^\times$ (Maschke's Thm)