(2). $H = \Bbbk[d]/(d^2)$, $K^\bullet, L^\bullet \in H\text{-gmod}$.

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{gmod}}}(K^\bullet, L^\bullet) = \frac{\{f : K^\bullet \longrightarrow L^\bullet \mid d\circ f = (-1)^{|f|} f\circ d\}}{\{f = d\cdot h = d\circ h - (-1)^{|h|} h\circ d\}}$$

$$= \mathrm{Hom}_{\mathrm{Com}(\Bbbk)}(K^\bullet, L^\bullet).$$

(3). $H = \Bbbk[\partial]/(\partial^p)$, $(\mathrm{char}(\Bbbk) = p > 0)$, $K^\bullet, L^\bullet \in H\text{-gmod}$.

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{gmod}}}(K^\bullet, L^\bullet) = \frac{\{f : K^\bullet \longrightarrow L^\bullet \mid \partial\circ f = f\circ\partial\}}{\{f = \partial^{p-1}\cdot h = \sum_{i=0}^{p-1} \partial^i\circ h\circ\partial^{p-1-i}\}}.$$

Def. $H=\Bbbk[\partial]/(\partial^p)$ $(\mathrm{char}(\Bbbk)=p>0)$

(1) The category of p-complexes $:= H\text{-}\mathrm{gmod}$.

(2) The homotopy category of p-complexes $:= H\text{-}\underline{\mathrm{gmod}}$.

Why care?

Lemma (Bernstein-Khovanov) $H\text{-}\underline{\mathrm{gmod}}$ is $\otimes$-triangulated. and

$$H\text{-}\underline{\mathrm{gmod}} \xrightarrow{K_0} \mathcal{O}_p$$

$$\begin{array}{ccccc} \oplus & \otimes & \longmapsto & + & \times \\ [1] & \{1\} & \longmapsto & -1 & \underline{q} \end{array}$$

Proof sketch. $\otimes$ descends to H-$\underline{\text{gmod}}$: $I \otimes M$ and $M \otimes I$ are injective if I is.

(1) $M \otimes (K \rightarrowtail I_K \twoheadrightarrow K[1]) = M \otimes K \rightarrowtail M \otimes I_K \twoheadrightarrow M \otimes K[1]$

$$\Longrightarrow (M \otimes K)[1] \cong (M \otimes K)[1].$$

(2) $M \otimes \left(\begin{tikzcd} K \arrow[r, "f"] \arrow[d, tail] \arrow[dr, phantom, "\#"] & L \arrow[d, tail] \\ I_K \arrow[r] \arrow[d] & C_f \arrow[d] \\ K[1] \arrow[r, equal] & K[1] \end{tikzcd} \right) \Longrightarrow M \otimes K \xrightarrow{\mathrm{Id} \otimes f} M \otimes L \longrightarrow M \otimes C_f \longrightarrow M \otimes K[1]$

remains a d.t.

Grothendieck group $K_0(H\text{-}\underline{gmod})$ is thus a ring, with

$$[K]\cdot[L] := [K\otimes L]$$

$$[\Bbbk] = 1 \qquad [\Bbbk\{i\}] = q^i$$

$K_0(H\text{-}\underline{gmod})$: generated by $\Bbbk\{i\}$, subject to the only relation

$$[H\{i\}] = q^i(1+q^2+\cdots+q^{p-1}) = 0$$

$$\Longrightarrow K_0(H\text{-}\underline{gmod}) \cong \frac{\mathbb{Z}[q,q^{-1}]}{(1+\cdots+q^{p-1})} = \mathcal{O}_p. \qquad \square$$

$$\boxed{H\text{-}\underline{\mathrm{gmod}} : \text{categorical } \mathcal{O}_p.}$$

Question: (1) How do we categorify modules over $\mathcal{O}_p$?

(2). How do we categorify algebras over $\mathcal{O}_p$ (e.g. $U_q(\mathfrak{g})$, $q^p=1$) ?

(3). How do we categorify $U_q(\mathfrak{g})$ modules and their tensor products ?

Categorification of $U_q(\mathfrak{sl}_2)$ at root of 1

Categorifying modules over $\mathcal{O}_p$

In usual homological algebra, modules over $\mathbb{Z}$ arise as $K_0(A\text{-mod})$ $K_0(Sh(X))$ etc. These categories A-mod and $Sh(X)$ can be described by differential graded algebras (DGA).

$$A = \bigoplus_{i \in \mathbb{Z}} A^i, \quad d_A : A^i \longrightarrow A^{i+1} \quad \text{s.t. } \forall a, b \in A$$

$$d_A^2(a) = 0,$$

$$d_A(ab) = d_A(a)b + (-1)^{|a|} a d_A(b).$$

Def. A p-DG algebra (A, ∂_A) is a graded algebra over a field of char $p>0$, equipped with a degree-one endomorphism ∂_A, s.t. $\forall a, b \in A$,

$$\partial_A^p(a) = 0,$$

$$\partial_A(ab) = \partial_A(a)b + a\partial_A(b)$$

A p-DG module (M, ∂_M) over a p-DG algebra (A, ∂_A) is a graded A-module M with a degree-one endomorphism ∂_M, s.t. $\forall a \in A$, $m \in M$,

$$\partial_M^p(m) = 0,$$

$$\partial_M(am) = \partial_A(a)m + a\partial_M(m).$$

Examples

(1). $\mathbb{k}[x]$: polynomial ring over $\mathbb{k}$, char$\mathbb{k} = p > 0$. Define a p-differential by setting $\partial(x) := x^2$ and extend it to $\mathbb{k}[x]$ by the Leibniz rule. Then $(\mathbb{k}[x], \partial)$ is a p-DGA.

(2) Consider $A = M_n(\mathbb{k})$, and J a (direct sum of) Jordan matrix of size(s) $\leq p$. Define $\partial_J(X) := [J, X]$. Then (A, ∂_J) is a p-DGA.

(3). $Sym_n := \mathbb{k}[x_1, \cdots, x_n]^{S_n}$, where $\partial(x_i) = x_i^2$, is a p-DGA.

	A : DGA :	A : p-DGA :
abelian	DG modules $/A$	p-DG modules $/A$
	$\Downarrow$ (mod nullhomotopy)	$\Downarrow$ (mod nullhomotopy)
triangulated	$\mathcal{C}(A,d)$: homotopy category	$\mathcal{C}(A,\partial)$: homotopy category
	$\Downarrow$ (invert qis)	$\Downarrow$ (invert qis)
triangulated	$\mathcal{D}(A,d)$: derived category	$\mathcal{D}(A,\partial)$: derived category

Generalization : Hopfological Algebra (Khovanov, Q.)

Thm. (Khovanov - Q) The derived category $\mathcal{D}(A,\partial)$ of p-DG modules over A admits a categorical action by $H-\underline{\text{gmod}}$:

$$
\begin{tikzcd}
H\text{-}\underline{\text{gmod}} \times \mathcal{D}^c(A) \arrow[r, "\otimes"] \arrow[d, Rightarrow] & \mathcal{D}^c(A) \arrow[d, Rightarrow] \\
\mathbb{O}_p \times K_0(A,\partial) \arrow[r, "\times"] & K_0(A,\partial)
\end{tikzcd}
$$

Why do we want to categorify $\mathcal{U}_q(sl_2)$?

- **Reshetikhin - Turaev - Witten :**

  $U_q(sl_2)$ is the quantized gauge group of 3d Chern-Simons theory.

  $(q^N = 1)$

- **Crane - Frenkel :**

  Categorify 3d Chern-Simons to a 4d-TQFT.

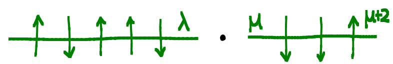
  $\mathcal{U}_q(sl_2)$: quantized 2-gauge group ?

Quantum $\mathfrak{sl}(2)$ at roots of unity

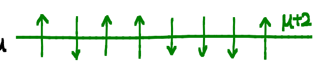
We are interested in the idempotented version of $U_q(\mathfrak{sl}_2)$. It is generated over $\mathbb{Z}[q, q^{-1}]$ by pictures of the form

$(\lambda \in \mathbb{Z})$

with the algebra structure

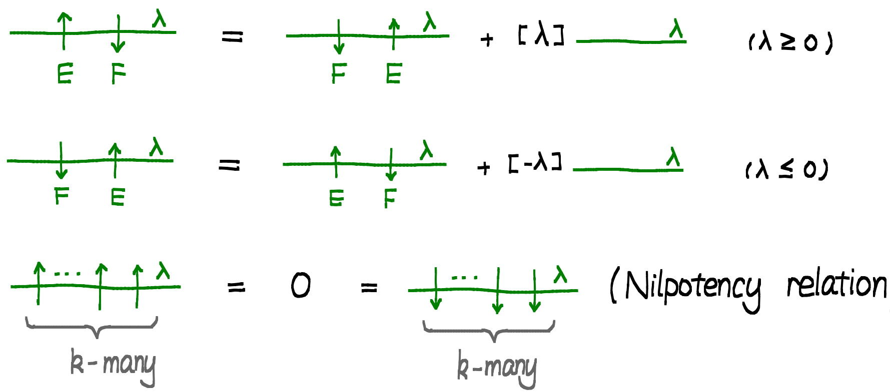


$=\ \delta_{\lambda\mu}$



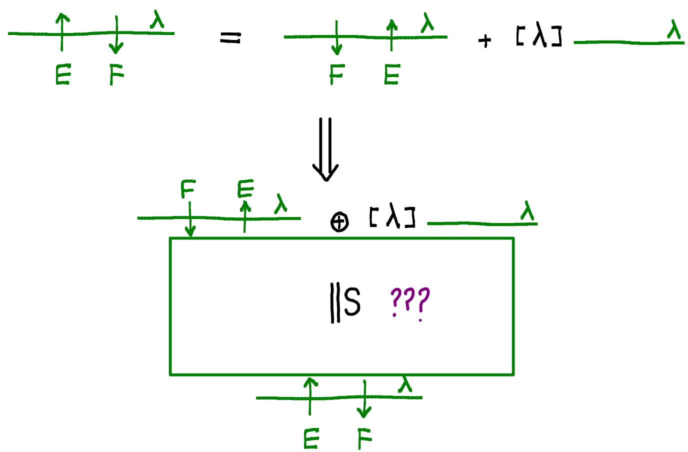
(etc)

Modulo relations (at a $2k$-th root of unity, k odd)



Categorification of $U_q(\mathfrak{sl}_2)$ (a la Khovanov-Lauda-Rouquier)

To categorify quantum groups, we want introduce an extra dimension ("time") to study "evolution" of quantum states:



The rough idea:

- 1-D pictures (horizontal slices) = (isomorphism classes of) projective modules $\mathcal{E}^{i_1}\mathcal{F}^{j_1}\ldots$

- 2-D pictures (vertical) = maps (evolution) between modules corresponding to 1-D slices

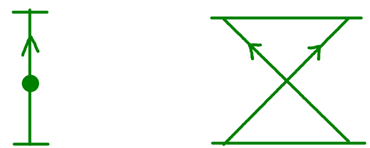
- Sum of 1-D pictures = symbol of direct sum of modules

- Equality of 1-D pictures = isomorphisms of modules.

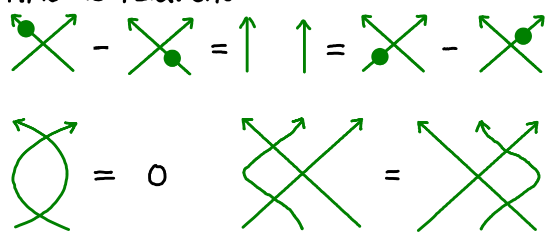
Below we present Lauda's diagrammatic calculus for $U_q(\mathfrak{sl}_2)$

- Maps just among E's (or F's).

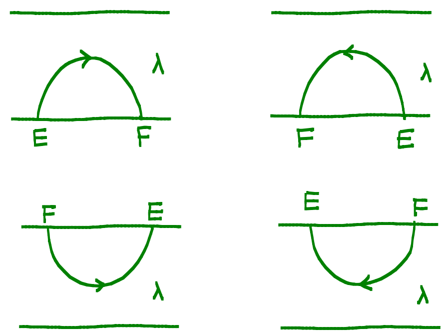
Generated by



subject to nilHecke relations:

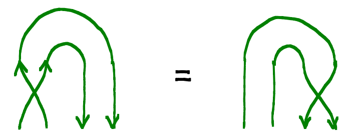


- To categorically connect E and F's, Lauda introduces cups and caps



Together with the nilHecke algebra generators, cups and caps satisfy certain relations

(i) Biadjointness   E.g.

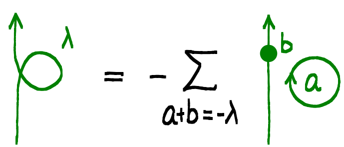


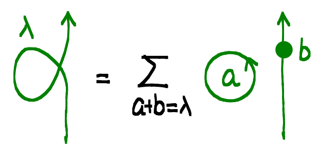
(ii) Bubble positivity : degrees of

must be ≥ 0.

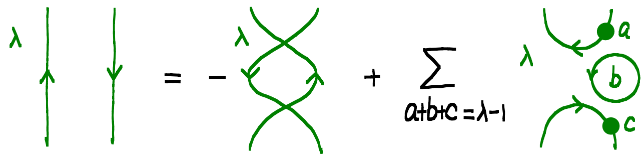
$k = m+1-\lambda \geq 0$ $\ell = m+1+\lambda \geq 0$

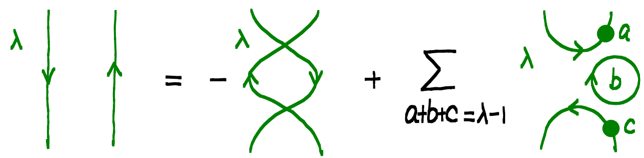
(iii) Reduction to bubbles





(U). Identity decomposition

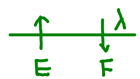




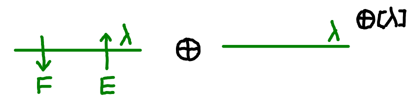
Thm. (Lauda) This graphical calculus, denoted $\mathcal{U}$, is non-degenerate and categorifies $\dot{U}_q(sl_2)$ at a generic q-value.

Rmk: Lauda's calculus is a 2-dim'l idempotented algebra, i.e. it has two compatible multiplication structures (vertical and horizontal). Such idempotented algebras are also known as a "2-category".

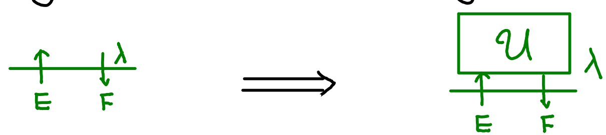
To illustrate the proof of Lauda's thm, let us consider how



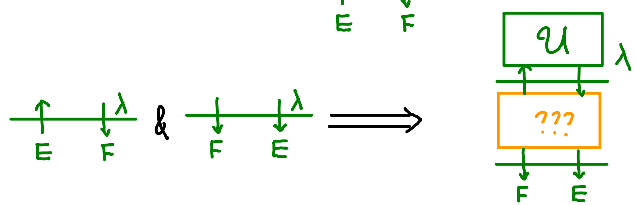
"evolves" into



According to the previous philosophy:



Maps between

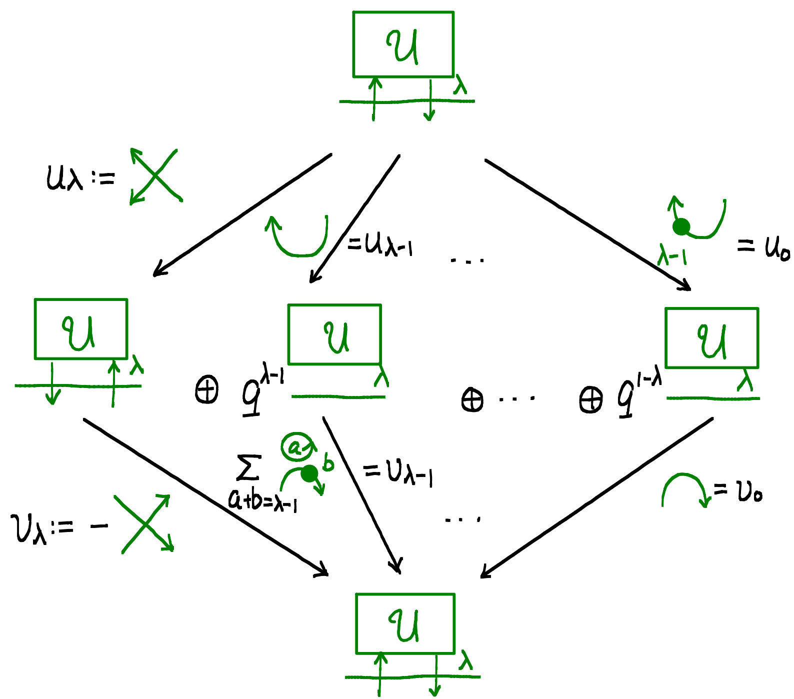


In general, to show that there is an isomorphism of A-modules $M \cong K \oplus L \iff \exists$ A-module maps

$$\begin{tikzcd} & M \arrow[dl, "u_1"'] \arrow[dr, "u_2"] & \\ K \arrow[dr, "v_1"'] & & L \arrow[dl, "v_2"] \\ & M & \end{tikzcd}$$

s.t. $1_M = v_1u_1 + v_2u_2$, $u_1v_1 = Id_K$, $u_2v_2 = Id_L$, $u_1v_2 = u_2v_1 = 0$

$\Longrightarrow$ v_1u_1, v_2u_2 are orthogonal idempotents in $End_A(M)$.



𝒰
λ
uλ :=
=uλ-1
...
λ-1
= u0
𝒰
𝒰
𝒰
⊕ q^{λ-1}
⊕ ...
⊕ q^{1-λ}
Σ
a+b=λ-1
a
b
=vλ-1
...
= v0
vλ:= −
𝒰

These elements $\{u_\lambda\}$, $\{v_\lambda\}$ satisfy

$$\begin{cases} v_i u_i = Id_i \\ v_i u_j = 0 \quad (i \neq j) \\ \sum u_i v_i = Id_{\mathcal{E}\mathcal{F}1_\lambda} \end{cases}$$

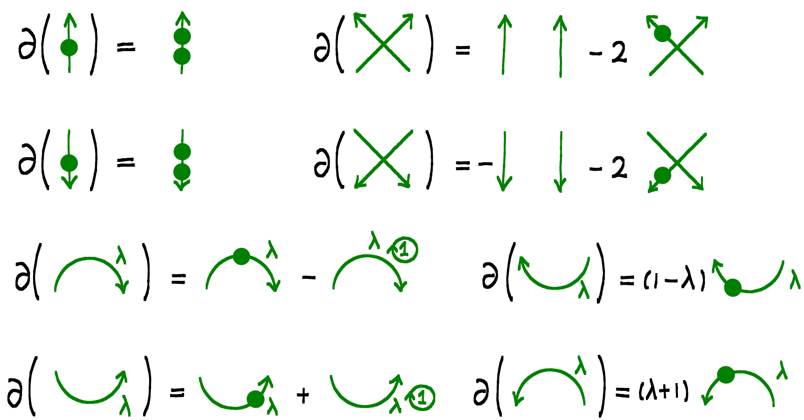
which follows from the identity decomposation relation.

Consequently $\{u_i v_i \mid i = 0, \cdots, \lambda\}$ form an orthogonal set of idempotents in $End_{\mathcal{U}}(\mathcal{E}\mathcal{F}1_\lambda)$

(Factorization of idempotents)

Enhancing $\mathcal{U}$ with a p-differential

Def. Let $(\mathcal{U}, \partial)$ be Lauda's 2-dimensional algebra equipped with the differential ∂-action on generators



Lemma. The above ∂ preserves all relations of $\mathcal{U}$, and it is p-nilpotent over a field of characteristic $p>0$.

Proof is a good exercise practicing with relations.

Thm. (Khovanov-Q., Elias-Q.) The derived module category $\mathcal{D}^b(\mathcal{U},\partial)$ categorifies $\dot{u}_{\mathbb{q}}(\mathfrak{sl}_2)$ at a p-th primitive root of 1:

$$K_0(\mathcal{U},\partial) \cong \dot{u}_{\mathbb{q}}(\mathfrak{sl}_2)$$

Decomposition v.s. filtration

In the realm of triangulated categories, direct sum decompositions are very rare.

Instead, a short exact sequence of p-DG $\mathcal{U}$-modules gives rise to a distinguished triangle in $\mathcal{D}(\mathcal{U},\partial)$.

$$0 \longrightarrow K \xrightarrow{v} M \xrightarrow{u} L \longrightarrow 0 \qquad \text{in} \quad (\mathcal{U},\partial)\text{-mod}$$

$$\Downarrow$$

$$K \xrightarrow{v} M \xrightarrow{u} L \xrightarrow{[1]} A[1] \qquad \text{in} \quad \mathcal{D}(\mathcal{U},\partial)$$

$$\Downarrow$$

$$[M] = [K] + [L] \qquad \text{in} \quad K_0(\mathcal{U},\partial)$$

More generally, a filtered p-DG module $(M, F^\bullet)$ presents M as a convolution (Postnikov tower) of $\mathrm{gr}F^\bullet$.

$$
\begin{tikzcd}
0 \arrow[r] & F^1 \arrow[r, hook] & F^2 \arrow[r, hook] \arrow[dl] & \cdots \arrow[r, hook] & F^n = M \arrow[dl] \\
& F^2/F^1 \arrow[ul] & F^3/F^2 \arrow[ul] \quad \cdots & F^n/F^{n-1} \arrow[ul] &
\end{tikzcd}
$$

$$\Longrightarrow \quad [M] = \textstyle\sum_i [F^i/F^{i-1}] \text{ in } K_0(\mathcal{U}, \partial).$$

Prop. Let $\{(u_i, v_i) \mid i \in I\}$ be factorization of idempotents in a p-DG algebra A. If there is a total ordering on I such that

$$\begin{cases} v_i \partial(u_i) = 0 \\ u_i \partial(v_i) \equiv 0 \pmod{\sum_{j<i} A u_j v_j} \end{cases}$$

Then if $\varepsilon = \sum_{i \in I} u_i v_i$, then the p-DG module $A\varepsilon$ admits a filtration $F^\bullet$ whose subquotients are isomorphic to $A v_i u_i$'s

Cor. In the situation of the Prop. $[A\varepsilon] = \sum_{i \in I} [A v_i u_i]$.

Proof of Prop.

Define $F^i := \sum_{j<i} Au_jv_j$. Then $F^i/F^{i-1} \cong Au_iv_i$.

(1). Inductively, F^i is ∂-closed, i.e. $\partial(u_iv_i) \in F^i$:

$$\partial(u_iv_i) = \partial(u_i)v_i + u_i\partial(v_i) = \partial(u_i)v_iu_iv_i + u_i\partial(v_i) \in Au_iv_i + F^{i-1} = F^i.$$

(2). Av_iu_i is ∂-closed:

Clear, since $v_iu_i = \mathrm{Id}_i$

(3). There is a p-DG module isomorphism:

$$\begin{tikzcd} (Au_iv_i \cong) F^i/F^{i-1} \arrow[r, bend left, "\cdot u_i"] & Av_iu_i \arrow[l, bend left, "\cdot v_i"] \end{tikzcd}$$

(ex). $\square$

Cor. Under the differential defined earlier on $\mathcal{U}$, there is a filtration on $\mathcal{E}\mathcal{F}\mathbb{1}_\lambda$