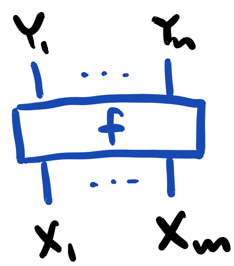
Webs (in type C)

based on joint work with L. Tatham and (time permitting) with E. Bodish, B. Elias, and L. Tatham

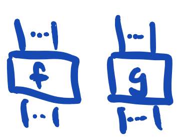
I. generalities

problem: find generators-and-relations presentations of (interesting) monoidal categories

recall: morphisms $f: \bigotimes_{i=1}^{m} X_i \longrightarrow \bigotimes_{j=1}^{n} Y_j$ in monoidal categories admit diagrammatic depictions:



example:

$f \otimes g \longleftrightarrow$ 

so we expect such presentations to consist of:

a) generating diagrams

b) local relations

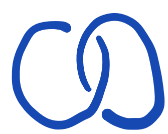
for today, the monoidal categories we find "interesting" are the categories $\mathrm{Rep}(U_q(\mathfrak{g}))$ of finite-dimensional representations of quantum groups associated to f.d. simple complex Lie algebras $\mathfrak{g}$

actually, we'll be interested in the full subcategory

$$\mathrm{FRep}(U_q(\mathfrak{g})) \subset \mathrm{Rep}(U_q(\mathfrak{g}))$$

tensor-generated by the fundamental representations. the latter can be recovered from the former by Karoubi completion.

<u>why do we care</u>?

 $\in \mathrm{End}(1)$

① explicit descriptions of associated <u>link invariants</u> / TQFT

② "foundation" for link homologies

③ reveal structural properties of these categories

II. what's known

$\mathfrak{g} = \mathfrak{sl}_2$: the following folk-theorem has origins in work of Rumer-Teller-Weyl, Temperley-Lieb, ...

thm: $\mathrm{FRep}(U_q(\mathfrak{sl}_2))$ is equivalent to the $\mathbb{C}(q)$-linear pivotal category freely generated by a single self-dual object, modulo the "circle relation"

$$\bigcirc = -[2]$$

this means:

- objects are $m \in \mathbb{N}$
- morphisms are generated by:

$$| \; , \; \cap \; , \; \cup$$

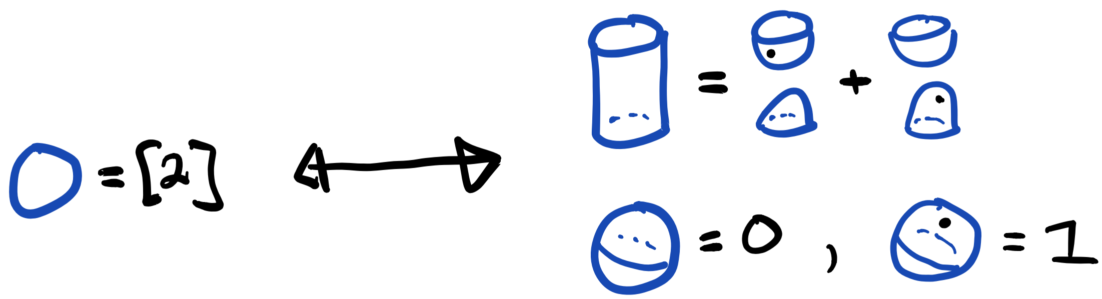
- relations are planar isotopy and the circle relation.

further, this category is ribbon, with braiding

$$\times = q^{1/2} \; | \, | \; + q^{-1/2} \; \substack{\cup \\ \cap}$$

$\rightsquigarrow$ Kauffman bracket description of the Jones polynomial $\rightsquigarrow$ Tait conj.

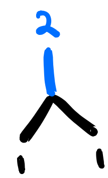
$\rightsquigarrow$ the "Bar-Natan 2-category" approach to Khovanov homology



<u>$\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4, \mathfrak{g}_2$</u>: Kuperberg proves an analogous result:

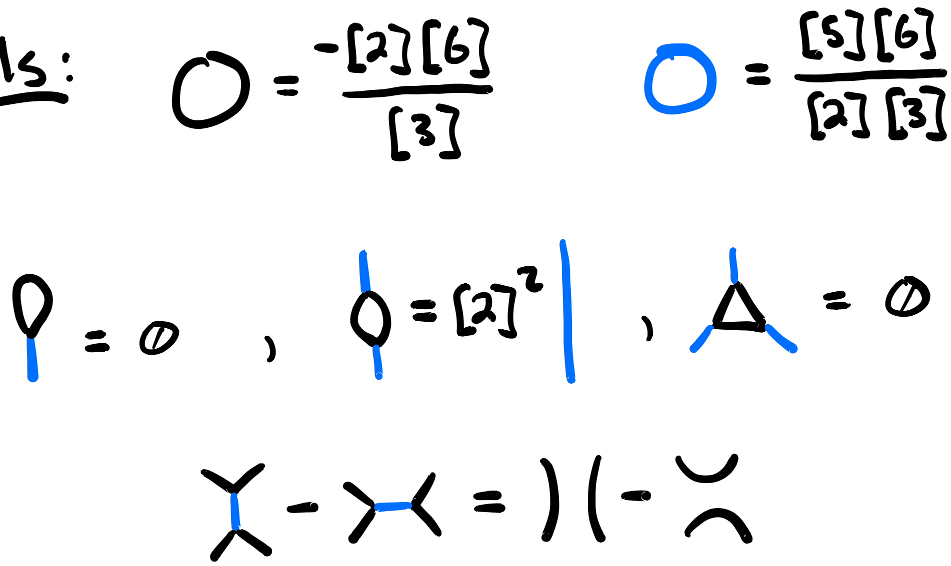
<u>thm</u> (Kuperberg): for $\mathfrak{g}$ rank 2, $\mathrm{FRep}(U_q(\mathfrak{g}))$ is equivalent to the pivotal category generated by (for $\mathfrak{g} \neq \mathfrak{sl}_3$, self-dual) objects $\{1, 2\}$ and ≤ 2 trivalent vertices, modulo ≤ 8 local relations.

<u>example</u>: $\mathfrak{g} = sp_4$

<u>gens</u>: 

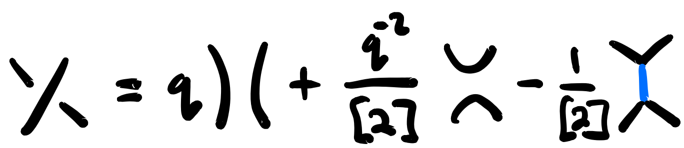
\+ those that are implied by "pivotal"

<u>rels</u>:



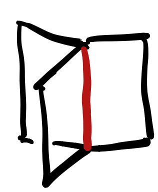
such graphs are called "webs"

again, we have an explicit formula for the braiding:



the $\mathfrak{g} = sl_3, \mathfrak{g}_2$ cases are similar.

$\xrightarrow[sl_3]{\text{for}}$ basis for Khovanov's sl_3 foam 2-category



Q: how are these results proved?

step 1: use Schur-Weyl duality (+ its relatives) to get a full, essentially surjective functor from the "free web category" to $\mathrm{FRep}(U_q(\mathfrak{g}))$.

step 2: use some Hom-space dimensions in $\mathrm{FRep}(U_q(\mathfrak{g}))$, and imposition of a compatible ribbon structure on webs, to determine some relations.

step 3: show that your relations allow webs to be simplified into a "special class" (e.g. those with certain types of faces), and do some (hard) work to count such webs that have a prescribed boundary.

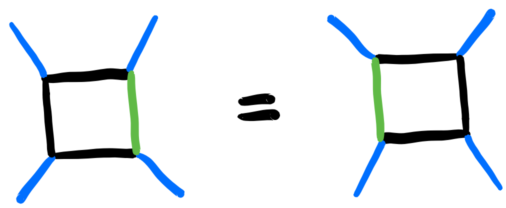
hard in two ways:

① how/why can you simplify? (in principle, hard even for closed webs)

② how to count?

$\mathfrak{g} = \mathfrak{sl}_n$: can try to repeat the above ...

... but you run into a "problem" right away at $\mathfrak{sl}_4$:



(doesn't reduce "face complexity")

the solution (Cautis - Kamnitzer - Morrison):

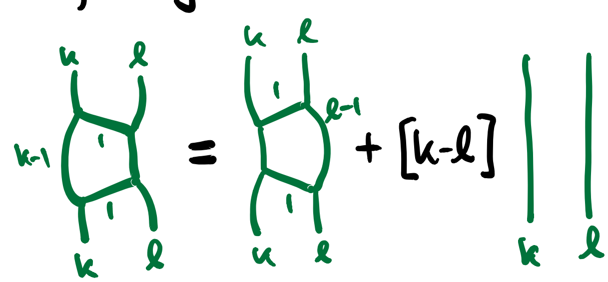
use skew Howe duality, i.e. the commuting actions of $U_q(\mathfrak{gl}_k) \circlearrowright \Lambda^\bullet(\mathbb{C}^k \otimes \mathbb{C}^n) \circlearrowleft U_q(\mathfrak{gl}_n)$

to get a functor

$$\dot{U}_q(\mathfrak{gl}_k) \longrightarrow \mathrm{FRep}(U_q(\mathfrak{sl}_n))$$

that gives web relations, e.g.

"$EF - FE = H$" $\longmapsto$



<u>and</u> a means to show you've found them all!

Idea (Sartori-Tubbenhauer): use skew Howe duality in other types to define web categories, and show they are equivalent to $\mathrm{FRep}(U_q(\mathfrak{g}))$ (or a close relative)

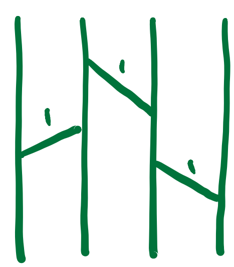
Problem: outside type A, Howe dualities don't quantize like you want them to...

another approach:

in type A, Howe duality has several implications for webs that can be exploited independently of their origins:

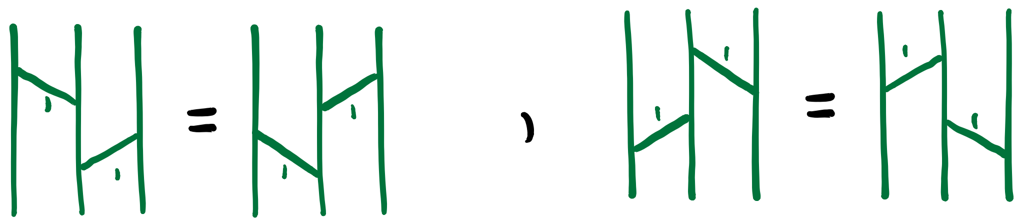
① it suggests considering webs that are in "ladder form":

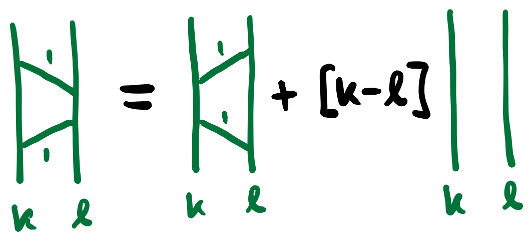
(all can be placed in this form)



$\longleftrightarrow E_2 F_1 E_3$

② it implies a PBW-like theorem for the rungs in the ladder:





(③ it also implies we should really be considering $\mathfrak{gl}_n$ ... maybe more on that later...)

①+② allow for proofs that:

a) all closed webs can be evaluated to scalars

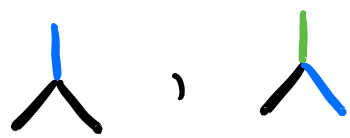
b) all Hom-spaces in $Web(\mathfrak{sl}_n)$ are finite-dimensional

c) $Tr(Web(\mathfrak{sl}_n)) \cong K_0(Rep(U_q(\mathfrak{sl}_n)))$

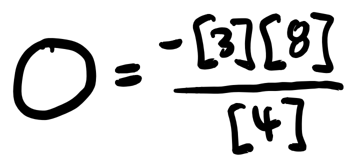
<u>without</u> reference to Howe duality.

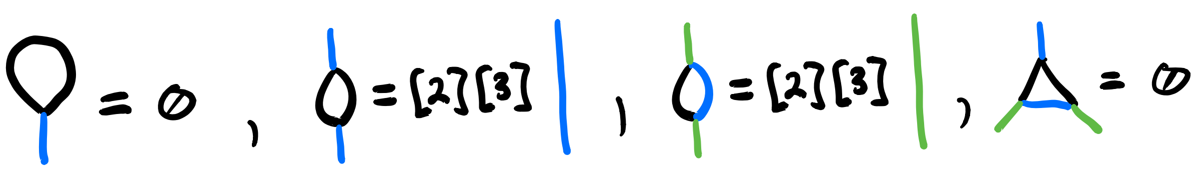
III. some new stuff

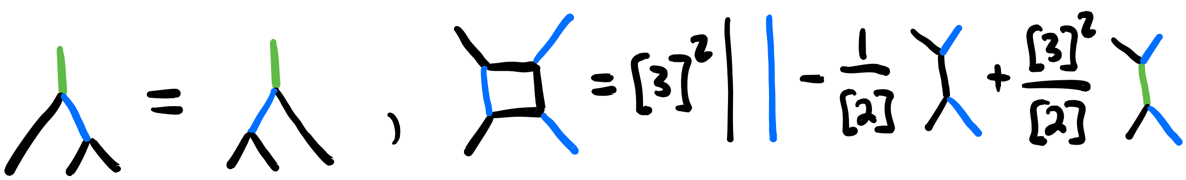
def: let $\mathrm{Web}(sp_6)$ be the $\mathbb{C}(q)$-linear pivotal category freely generated by self-dual objects $\{1, 2, 3\}$ and morphisms

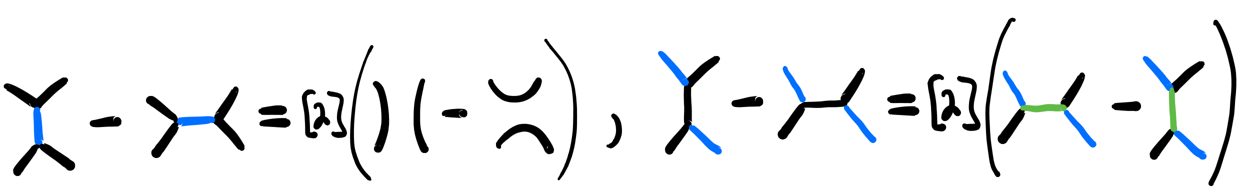


modulo the relations:









in joint work with L. Tatham, we show...

thm 1: $\mathrm{Web}(\mathfrak{sp}_6)$ is ribbon, and there is a full, essentially surjective, braided, monoidal functor $\mathrm{Web}(\mathfrak{sp}_6) \xrightarrow{\Psi} \mathrm{FRep}(U_q(\mathfrak{sp}_6))$

thm 2: $\mathrm{End}_{\mathrm{Web}(\mathfrak{sp}_6)}(\emptyset) \cong \mathbb{C}(q)$, i.e. all closed webs evaluate to scalars. moreover, all Hom-spaces in $\mathrm{Web}(\mathfrak{sp}_6)$ are finite-dimensional

thm 3: the functor Ψ induces an isomorphism $\mathrm{Tr}(\mathrm{Web}(\mathfrak{sp}_6)) \cong K_0^{\mathbb{C}}(\mathrm{Rep}(U_q(\mathfrak{sp}_6)))$ of commutative $\mathbb{C}(q)$-algebras.

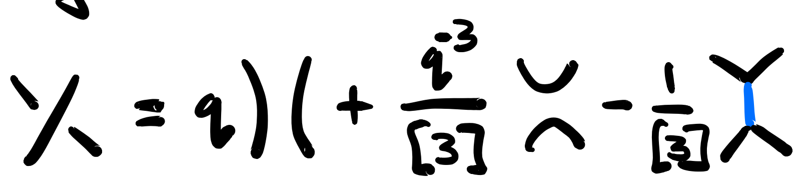
conj: Ψ is faithful, thus $\mathrm{Web}(\mathfrak{sp}_6) \cong \mathrm{FRep}(U_q(\mathfrak{sp}_6))$

resolution of this in progress with Bodish, Elias, Tatham some remarks:

① thm 1 follows via the "easy" steps 1 and 2 from before.

② thms 2 and 3 follow by an application of "rung PBW" to a category of "$\mathfrak{gsp}_6$ ladder webs." this is motivated by observation ③ above about ladders in type A, and by the Lie group GSp_6.

③ thms 1,2,3 and the formula for the braiding:



give a Kauffmanesque formulation of the $U_q(sp_6)$ link polynomial.

$\Rightarrow$ Web(sp_6) is "good enough" for link invariants