Howe to translate Gelfand-Tsetlin

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Definition

The category $\mathcal{S}_c$ of **Soergel bimodules** (for $\mathfrak{g}(n)$) is the Karoubi envelope of the monoidal subcategory of graded bimodules over $S = \mathbb{C}[x_1, \ldots, x_n]$ generated by the bimodules $B_i = S \otimes_{S^{x_i}} S$.

Different people appreciate this category because of the multifarious ways its appears in mathematics.

1. combinatorially either as above, or in terms of Soergel calculus (which I won’t describe in detail here).
2. representation theoretically in terms of translation and projective functors.
3. geometrically in terms of perverse sheaves on flag varieties.

Let me give a quick sketch of these constructions.
Basic notation for $\mathfrak{g} = \mathfrak{gl}_n$:

- A weight is given by an $n$-tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Dominant integral if $\lambda_i \in \mathbb{Z}$, and $\lambda_1 \leq \cdots \leq \lambda_n$.

- The center $Z_n = Z(U(\mathfrak{g}))$ is isomorphic to $\mathbb{C}[z_1, \ldots, z_n]^{S_n}$ with $f(z_1, \ldots, z_n)$ acting by $f(\lambda_1 + 1, \ldots, \lambda_n + n)$ on the Verma $M(\lambda)$.

- Let $I_0 = \{f \in Z | f(1, \ldots, n) = 0\}$ be the annihilator of the trivial module, and $\hat{Z}_n$ be the completion of $Z_n$ at this ideal. This is isomorphic to the completion $\hat{S} \cong \hat{Z}_n$ via the map $x_i \mapsto z_i - i$.

- We have an inclusion $\iota_k : Z_k \hookrightarrow U(\mathfrak{gl}_k) \hookrightarrow U(\mathfrak{gl}_n)$. The subring $\Gamma$ generated $\iota_k(Z_k)$ for $k = 1, \ldots, n$ is called the **Gelfand-Tsetlin** subalgebra.
Part I

Break time!

Part II

Background on $\mathfrak{gl}_n$

Definition

A module $M$ over $U(\mathfrak{gl}_n)$ is **Gelfand-Tsetlin** if it is $\Gamma$-locally finite, i.e. for any $m \in M$, we have $\dim(\Gamma m) < \infty$.

Finite dimensional modules are obviously Gelfand-Tsetlin, as are Verma modules for all Borels containing torus (so all objects in categories $\mathcal{O}$).

Questions:

- What are the simple Gelfand-Tsetlin modules? (Hard; only solved in 2018 by KTWWY.)
- How does tensor product with finite dimensional modules act on GT modules? (Less hard, I’ll explain today.)
For $\chi \in \text{MaxSpec}(\mathbb{Z}_n)$, let $C_\chi$ be the subcategory of $U(\mathfrak{gl}_n)$ modules where a power of $I_\chi$ acts trivially, and $\text{pr}_\chi : \mathfrak{g}\text{-mod} \to C_\chi$ be functor of the largest subobject in this category.

For any finite dimensional $\mathfrak{g}$-module $U$, we, have a functor $\text{pr}_{\chi'}(U \otimes -) : C_\chi \to C_{\chi'}$. The category $\mathcal{S}_r(\chi, \chi')$ of projective functors are sums of summands of these.

**Theorem (Bernstein-Gelfand, Soergel)**

*There's a tensor equivalence between $\mathcal{S}_r = \mathcal{S}_r(0, 0)$ of projective functors $C_0 \to C_0$ and the category $\mathcal{F}_c$ of completed (ungraded) Soergel $\widehat{\mathbb{Z}}_n - \widehat{\mathbb{Z}}_n$-bimodules.*

The bimodule $S^iS_S$ corresponds to translation onto a wall with $x_i = x_{i+1}$, and $S^iS^i_S$ to translation off.
On the other hand, we can also interpret these bimodules geometrically. Let

\[ G = GL_n \quad B = \begin{bmatrix} \ast & \ast & \ast \\ \ast & 0 & \ast \\ \ast & \ast & \ast \end{bmatrix} \quad P_i = \begin{bmatrix} \ast \\ \ast \\ \ast \end{bmatrix} \]

Soergel bimodules are an algebraic reflection of the geometry of the double coset space \( B \backslash G / B \). This space has a category \( \mathcal{S}_g \) of sums of shifts of semi-simple perverse sheaves inside the derived category, which is monoidal under convolution.

**Theorem (Soergel)**

The pushforward \( B \backslash G / B \to B \backslash * / B \) induces a monoidal equivalence \( \mathcal{S}_g \to \mathcal{S}_c \), matching homological grading of perverse sheaves to internal grading of Soergel bimodules, sending \( IC(P_i) \) to \( S \otimes_{S^{*i}} S \).

These results together are the key to the self-Koszul duality of category \( \mathcal{O} \) (since simple perverse sheaves correspond to projective functors).
So we have a sequence of functors of additive categories:

\[ \mathcal{S}_g \to \mathcal{S}_c \to \mathcal{S}_r \]

with the first an equivalence, and the second an equivalence after completion and forgetting gradings.

We can extend this to the singular case as well. For each \( \chi \in \text{MaxSpec}_\mathbb{Z}(\mathbb{Z}_n) \), we have a parabolic \( P_\chi \), corresponding invariant ring \( S^{W_\chi} \), and have equivalences:

\[ \mathcal{S}_g(\chi, \chi') = \text{Perv}(P_\chi \backslash G/P_{\chi'}) \to \mathcal{S}_c(\chi, \chi') \to \mathcal{S}_r(\chi, \chi') \]

The resulting 2-category is a quotient of categorified \( \mathfrak{sl}_\infty \).
Let’s compare this with Joel’s talk. Let $\mathcal{O}_\chi \subset \mathcal{C}_\chi$ is the category of weight modules which are $U(b)$ locally finite.

**Theorem (Joel’s talk)**

We have an isomorphism

$$\bigoplus_{\chi \in \text{MaxSpec}_\mathbb{Z}(\Gamma)} K^0(\mathcal{O}_\chi) \cong \text{Sym}^n(\mathbb{C}^\infty \otimes \mathbb{C}^n)_0$$

controls $\chi$

$$K^0(G\mathcal{F}_\chi) \cong U(\mathfrak{sl}_n^+) \otimes \text{Sym}^n(\mathbb{C}^\infty)$$

to the 0-weight space for the $\mathfrak{sl}_n$-action on the RHS.

Joel’s functors change the underlying algebra (to a finite $W$-algebra or worse). On the other hand, the projective functors give the commuting Howe dual action of $\mathfrak{sl}_\infty$. So, my talk is the Howe dual of Joel’s.

$$U(\mathfrak{sl}_n^+) \otimes (\mathbb{C}^n)$$
In [KTWWY], we didn’t just identify category $\mathcal{O}$. We found the whole category of Gelfand-Tsetlin modules $\mathcal{GT}_\chi$.

Let $\tilde{T}_\chi$ denote the KLRW algebra for the Dynkin diagram $1 - 2 - 3 - \cdots - (n-1)$, with

- red strands with $x$-values given by the entries of $\chi$ (when $\chi$ is singular, we get thick strands from the repeats), all labeled by the appropriate multiple of the fundamental weight $\omega_{n-1}$.
- $k$ black strands with label $k$ for all $k = 1, \ldots, n-1$.
- dots on both red and black strands.
Theorem

The category $\mathcal{GT}_\chi$ is equivalent to the category of weakly-graded finite dimensional $\tilde{T}^{\chi}$-modules.

Under this equivalence, the images of the obvious idempotents in $\tilde{T}^{\chi}$ match with the weight spaces for elements of $\text{MaxSpec}(\Gamma)$.

- to get $\mathcal{O}_\chi$, kill idempotents corresponding to weight spaces not allowed in category $\mathcal{O}$.
- red dots = nilpotent part of $Z_n$ action.
- sum of black dots on strands with label $i =$ nilpotent part of $U(\mathfrak{h})$-action.

So, we have a Soergel bimodule action on $\tilde{T}^0$-modules, and a categorical $\mathfrak{sl}_\infty$ action on all $\chi$’s together. How can we describe it in these combinatorial terms?
### The talk thus far

**Part I**

**Break time!**

**Part II**

The talk thus far involves various mathematical concepts and theories, including geometry, combinatorics, and representation theory. Here’s a breakdown:

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<td>Perverse sheaves on $P_{\chi} \backslash G/P_{\chi'}$</td>
<td>$S_{W_{\chi}} - S_{W_{\chi'}}$ Soergel bimodules</td>
<td>Projective functors $C_{\chi} \rightarrow C_{\chi'}$</td>
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<tr>
<td>??</td>
<td>??</td>
<td>Usual action</td>
</tr>
<tr>
<td>??</td>
<td>$\tilde{T}_{\chi}$-modules</td>
<td>Gelfand-Tsetlin modules $G_{\mathcal{T}_{\chi}}$</td>
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Translation onto/off of the wall corresponds to “splitter bimodules” from Khovanov-Lauda-Sussan-Yonezawa.

Key observation of the proof:

\[
U(\mathfrak{gl}_n) \otimes \mathbb{C}^n \cong U(\mathfrak{gl}_n)E_n U(\mathfrak{gl}_n) \subset U(\mathfrak{gl}_{n+1})
\]

**Theorem (W.)**

*The monoidal category \( \mathcal{S}_c \) acts on \( \tilde{T}^0 \)-modules via the KLSY bimodules.*

Note, this shows why red-dotting was needed: projective functors don’t preserve semi-simple action of the center.
Of course, this only covers very specific number of black strands, and in particular, only a few of the $\mathfrak{sl}_2$ cases KLSY consider.

One fix: generalize $U(\mathfrak{g}l_n)$ to other Coulomb branches.

- this includes finite $W$-algebras if
  \[ v_1 \leq v_2 - v_1 \leq \cdots \leq v_{n-1} - v_{n-2} \leq n - v_{n-1}. \]
- other weirder stuff in other cases.

A bit tricky to write out details of, though.

Solution I prefer: get that last corner of my summary page, geometry.
Choose an $m$-tuple of integers $(v_1, \ldots, v_{m-1}, v_m = n)$. Let

$$V = \text{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \oplus \text{Hom}(\mathbb{C}^{v_2}, \mathbb{C}^{v_3}) \oplus \cdots \oplus \text{Hom}(\mathbb{C}^{v_{m-2}}, \mathbb{C}^{v_{m-1}}) \oplus \text{Hom}(\mathbb{C}^{v_{m-1}}, \mathbb{C}^n)$$

$$H_0 = GL(v_1) \times GL(v_2) \times \cdots \times GL(v_{m-1})$$

Almost a moduli of quiver reps, but note that $B$ instead of a $G$.

**Theorem (Guan-W.)**

*The $H$-orbits on $V$ are classified by ways of writing $\mathbf{v}$ as a sum of positive roots, with a choice of order on the roots of type $(0, \ldots, 0, 1, \ldots, 1)$ appearing.*
Let $V_{\text{inj}} \subset V$ be the subspace where all the maps $f_i: \mathbb{C}^{v_i} \rightarrow \mathbb{C}^{v_i+1}$ are injective.

In the case $v = (1, 2, \ldots, n)$, we have a close relationship to the flag variety.

**Lemma**

We have a $G$-equivariant isomorphism $V_{\text{inj}}/H_0 \cong B\backslash G$ by thinking of

$$\text{im}(f_{n-1}) \supset \text{im}(f_{n-1}f_{n-2}) \supset \cdots \supset \text{im}(f_{n-1} \cdots f_1)$$

as a flag.
Part I
Connection to quivers

So, we have a category of sums of shifts of semi-simple perverse sheaves $\text{Perv}(V/H)$.

**Theorem**

*The category of $\text{Perv}(V/H)$ carries an action by convolution of $\mathcal{S}_g = \text{Perv}(B\backslash G/B)$ via convolution.*

This is a general observation about spaces with a $G$-action that we restrict to the action of $B$.

If we let $H_i = H_0 \times P_i$, then the action of $\text{IC}(P_i)$ is pushing and pulling on the map $V/H \to V/H_i$.

As usual, we can generalize to the singular case by considering $\text{Perv}(V/H_\chi)$ for $H_\chi = H_0 \times P_\chi$. 

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As you might expect, this matches with the other categories with Soergel actions:

**Theorem**

The category $\text{Perv}(V/H_\chi)$ is equivalent to the category of graded projective $\tilde{T}_\chi$-modules. This intertwines the Soergel action on $\tilde{T}_0$-modules and $\mathfrak{sl}_\infty$-action on all $\chi$ with that by KLSY bimodules.

This is the easiest way to prove that such an action exists. Of course, you have to work algebraically if want to do $p$-DG (for now).

Restricting to $V_{\text{inj}}$ has effect of passing to $\emptyset$, back to Joel’s talk.

Proof analogous to CVD.
# The whole talk

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Thanks for listening.