Miraculous Treumann-Smith Theory and Geometric Satake (w. S. Riche) ①

Beilinson-Bernstein Localization

$\mathfrak{g}/\mathbb{C}$ complex semi-simple
$\rightsquigarrow$ flag variety $\mathcal{B}$

$$\mathcal{D}_{\mathcal{B}}\text{-mod} \xrightarrow[\sim]{\Gamma} U(\mathfrak{g})/(Z_+)\text{-mod}$$

→ direct relationship

→ block by block

→ Kazhdan-Lusztig conjecture manifest

E.g. $\mathcal{D}_{\mathbb{P}^1_{\mathbb{C}}}\text{-mod} \simeq \dfrac{U(\mathfrak{sl}_2)}{\langle 4fe+2h+h^2\rangle}\text{-mod}$ Casimir

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Geometric Satake Equivalence

G/k split reductive group
$\rightsquigarrow$ $\mathcal{G}r$ complex affine Grassmannian for Langlands dual group $G^\vee/\mathbb{C}$.

$$(\mathrm{Rep}\, G, \otimes_k) \xrightarrow{\sim} (\mathrm{Perv}_{G^\vee[[t]]}(\mathcal{G}r, k), *)$$

→ relationship indirect

→ block decomposition opaque

→ Lusztig conj ??

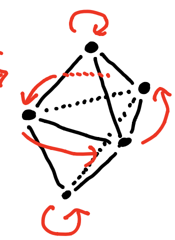
Dream: geometric understanding of Rep G passing through geometric Satake.

Equivariant Localisation

Principle: $\Gamma \circlearrowright X$, relate "stuff on X" to "stuff on X^{Γ}".

↳ integrals, Euler char., cohomology, K-theory, ...

Example. $\mathbb{Z}/p\mathbb{Z} \circlearrowright$ finite CW complex X

$\Gamma = \mathbb{Z}/3\mathbb{Z} \circlearrowright$ 

$$\boxed{\chi(X) = \chi(X^{\mathbb{Z}/p\mathbb{Z}}) \bmod p}$$

Homological perspective $\qquad \Gamma = \mathbb{Z}/p\mathbb{Z}$

$\mathbb{F}_p$-chains of $X^{\Gamma} \hookrightarrow \mathbb{F}_p$-chains on $X \twoheadrightarrow$ perfect complex of $\mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}]$-modules.

(bounded complex of free/proj. $\mathbb{F}_p[\Gamma]$-modules.)

$K^b(\mathbb{F}_p\Gamma\text{-mod}) / (\text{perfect complexes})$ $\qquad$ images agree.

Smith–Treumann Theory

X complex algebraic variety

$\Gamma = \mathbb{Z}/p\mathbb{Z} \circlearrowright X$

$$\begin{tikzcd} X \\ X^{\Gamma} \arrow[u, hook, "i"'] \end{tikzcd}$$

$$\begin{tikzcd} & D^b_{\Gamma}(X, \mathbb{F}_p) \arrow[d, shift right, "i^!"'] \arrow[d, shift left, "i^*"] \arrow[dl, "i^{!*}"', "\text{“Smith restriction”}"] \\ \mathrm{Sm}(X^{\Gamma}) & D^b_{\Gamma}(X^{\Gamma}, \mathbb{F}_p) \arrow[l, "\text{quotient}"] \end{tikzcd}$$

$D^b_{\Gamma}(X, \mathbb{F}_p)$ = bounded derived category of Γ-equivariant $\mathbb{F}_p$-sheaves

$D^b_{\Gamma}(X^{\Gamma}, \mathbb{F}_p) \simeq D^b(X^{\Gamma}, \mathbb{F}_p[\Gamma])$

↖ action is trivial

Key lemma (Treumann): $\quad \mathrm{Cone}(i^! \to i^*)$ is perfect.

$$\boxed{\mathrm{Sm}(X^{\Gamma}) = D^b_{\Gamma}(X^{\Gamma}, \mathbb{F}_p) \Big/ \left(\begin{array}{l} \text{complexes w/} \\ \text{perfect stalks} \end{array} \right)}$$

Meta-theorem (Treumann): $\quad i^{!*}$ commutes with everything.

Useful variant (Leslie-Lonergan, W.)

Assume: $\mathbb{C}^\times \curvearrowright X$, $\quad \mu_p \subset \mathbb{C}^\times$

"action extends"

$$\boxed{\mathrm{Sm}_{\mathbb{C}^\times}(X^{\mu_p}, \mathbb{F}_p) := D^b_{\mathbb{C}^\times}(X^{\mu_p}, \mathbb{F}_p) \Big/ \Big\langle \begin{array}{c} \mathcal{F} \text{ whose} \\ \text{restriction to} \\ \mu_p \subset \mathbb{C}^\times \text{ is} \\ \text{perfect} \end{array} \Big\rangle}$$

"Equivariant Smith quotient"

Example: $\mathbb{C}^\times \curvearrowright \mathrm{pt}$:

$$\begin{tikzcd} D^b_{\mathbb{C}^\times}(\mathrm{pt}) \arrow[r, "\sim"] \arrow[d] & \mathrm{dgDer}\, \mathbb{F}_p[x] \arrow[d] \\ \mathrm{Sm}_{\mathbb{C}^\times}(\mathrm{pt}) \arrow[r, "\sim"] & \mathrm{dgDer}\, \mathbb{F}_p[x^{\pm 1}] \end{tikzcd}$$

deg 2 (x)

$$\Rightarrow \mathrm{Ext}^i_{\mathrm{Sm}_{\mathbb{C}^\times}(\mathrm{pt})}(\mathbb{F}_p, \mathbb{F}_p) = \begin{cases} 0 & i \text{ odd}, \\ \mathbb{F}_p & i \text{ even}. \end{cases}$$

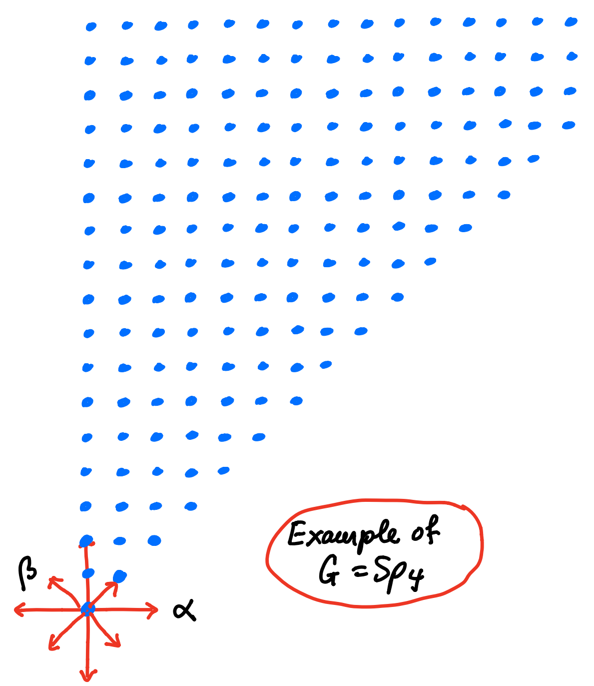
$\leadsto$ parity sheaves make sense

(L.L.)

Algebraic Representations

G/k split reductive group
$\cup$
B Borel subalgebra
$\cup$
T maximal torus

$\mathfrak{X}$ character lattice
$\cup$
$\mathfrak{X}_+$ dominant weights



Example of $G = Sp_4$

Chevalley Theorem

$$\left\{ \begin{array}{c} \text{simple} \\ G\text{-modules} \end{array} \right\} \Big/ \cong \quad = \quad \mathfrak{X}_+$$

$$\begin{array}{ccc} \Psi & & \Psi \\ L_\lambda & \longleftarrow & \lambda \end{array}$$

highest weight λ

p = char k

Linkage Principle

affine Weyl group (of DUAL group) $\quad W = W_f \ltimes \mathbb{Z}\Phi$

$$\cup$$

$$W_p = W_f \ltimes p\mathbb{Z}\Phi$$

Dot action: $\quad x\cdot\lambda = x(\lambda+\rho)-\rho$

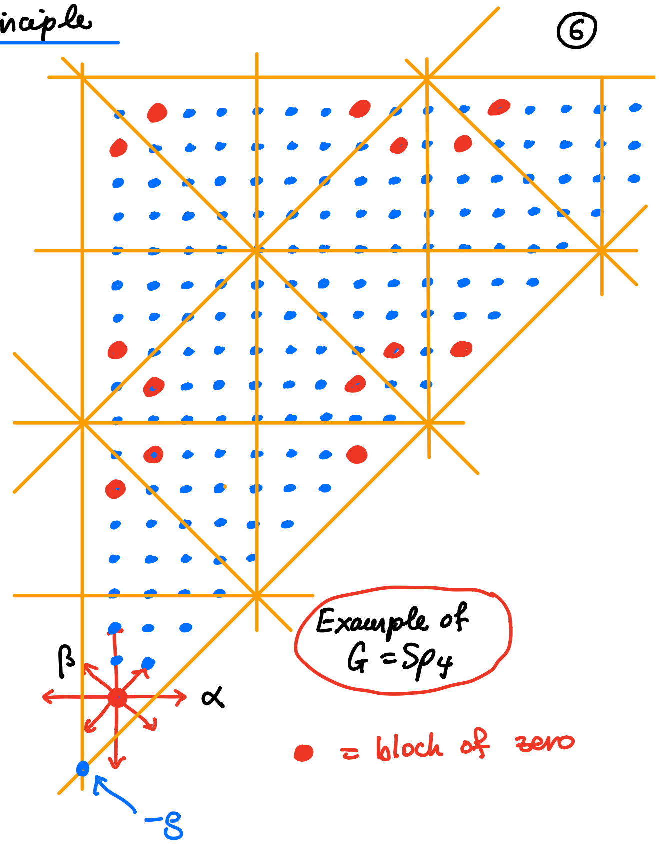
$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$$

Linkage Principle

$$\mathrm{Rep}\,G = \bigoplus_{\gamma\in \mathscr{X}/(W_p\cdot)} \mathrm{Rep}_\gamma G$$

where

$$\mathrm{Rep}_\gamma G = \langle L_\lambda \mid \lambda\in\gamma\cap \mathscr{X}_+\rangle$$

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Closed Caption
Breakout Rooms
Reactions
More
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⑦

Affine Grassmannian

$G \rightsquigarrow G^\vee / \mathbb{C}$
Langlands dual group

$$\mathcal{G}r_{G^\vee} = G^\vee(\mathbb{C}((t))) / G^\vee(\mathbb{C}[[t]])$$

(∞-dim projective variety / ind. projective ind. scheme)

Geometric Satake Equivalence

$$(\mathrm{Rep}\, G, \otimes) \xrightarrow{\sim} (\mathrm{Perv}_{G^\vee(\mathbb{C}[[t]])}(\mathcal{G}r, k), *)$$

$$\begin{array}{ccc} \cup & & \cup \\ L_\lambda & \longleftrightarrow & IC_\lambda \end{array}$$

Remark: Not a derived equivalence: $D^b(\mathrm{Rep}\, G) \not\simeq D^b_{G^\vee(\mathbb{C}[[t]])}(\mathcal{G}r, k)$

⑨

Loop Rotation

Natural action of $\mathbb{C}^\times$ on $\mathbb{C}((t))$ induced by $t \mapsto zt$ $\rightsquigarrow$ loop rotation action $\mathbb{C}^\times \curvearrowright \mathcal{G}r$.

Observation
(learnt from R. Bezrukavnikov outside Kendall/MIT in 2016)

$$\boxed{(\mathcal{G}r)^{\mu_p} = \bigsqcup_{\gamma \in \mathcal{X}/W_p} \mathcal{G}r_{G^\vee, (\gamma)}}$$

Moreover, each component $\mathcal{G}r^\gamma_{G^\vee}$ is a partial affine flag variety for $G^\vee((t^p))$.

Examples:

① $\mathcal{G}r^0_{G^\vee} = G^\vee((t^p))/G^\vee[[t^p]]$
"thin Grassmannian"

② γ p-regular, $\mathcal{G}r_{G^\vee,(\gamma)} \simeq$ affine flag variety.

Compare:

Linkage principle

$$\operatorname{Rep} G = \bigoplus_{\gamma \in \mathfrak{X}/(W_p \cdot)} \operatorname{Rep}_\gamma G$$

μ_p fixed points

$$(\mathcal{G}r)^{\mu_p} = \bigsqcup_{\gamma \in \mathfrak{X}/W_p} \mathcal{G}r^{\gamma}_{G^\vee}$$

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Compare:

Linkage principle

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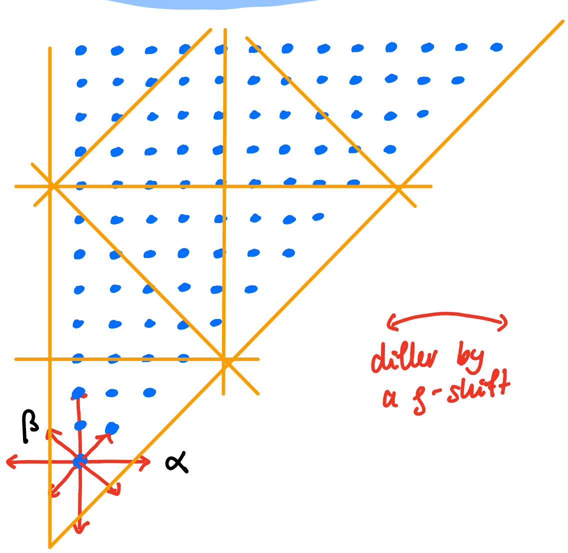
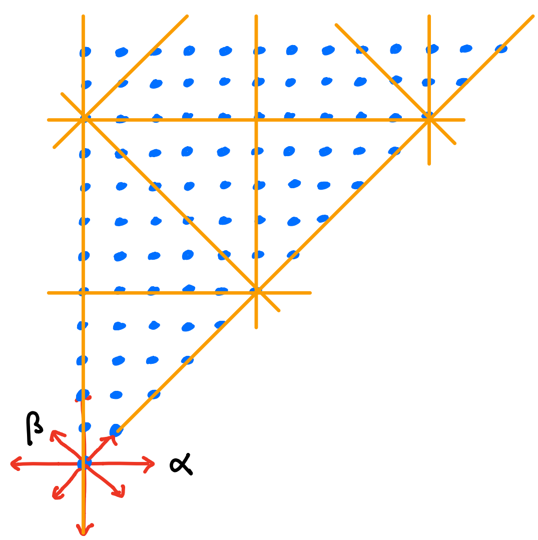


differ by a ρ-shift



Bezrukavnikov – Mirković – Gaitsgory – Riche – Rider Theorem

$$IC_\lambda \in \mathrm{Perv}_{G^\vee[[t]]}(Gr, k) \xrightarrow{\sim} \mathrm{Perv}_{IW}(Gr, k) \ni IC_{\lambda+\varsigma}$$

"Satake category" "Iwahori-Whittaker model"

1. inspired by Whittaker model in representation theory of p-adic groups
2. needs Artin-Schreier local system
   $\mathbb{C} \rightsquigarrow \overline{\mathbb{F}}_\ell$, $\ell \neq p$.
3. extends to derived categories
   $$D^b(\mathrm{Rep}\, G) \simeq D^b_{IW}(Gr, k)$$
4. Components of μ_p-fixed points precisely match linkage principle!

Main Theorem: (w/ Riche)

$$\begin{tikzcd}
D^b_{IW,\mathbb{G}_m}(Gr, k) \arrow[r, "i^{!*}"] & \mathrm{Sm}_{IW,\mathbb{G}_m}((Gr_{G^\vee})^{\mu_p}, k) \\
\mathrm{Parity}^{even}_{IW,\mathbb{G}_m}(Gr, k) \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] \arrow[d, phantom, "\wr\wr"] & \mathrm{Sm}\,\mathrm{Parity}^{even}_{IW,\mathbb{G}_m}((Gr)^{\mu_p}, k) \arrow[u, phantom, "\cup"] \\
\text{tilting modules for } G &
\end{tikzcd}$$

Corollary: $\mathrm{Hom}(T_\lambda, T_\mu) \neq 0 \implies$ $\lambda+\varsigma$ and $\mu+\varsigma$ lie in same component of $(Gr)^{\mu_p}$.

$\implies$ geometric proof of linkage principle.

Character Formulas

→ first proof of Lusztig's character formula for $p >> 0$ via geometric Satake.

→ **Thm:** $\forall$ blocks $\forall$ p!

$$\mathrm{ch}\, T_{x \cdot \mu} = \sum {}^{p}n_{y,x}(1)\ \mathrm{ch}\, \Delta_{y \cdot \mu}$$

Conjectured by Riche-W. in 2016.

Proved by Achar-Makisumi-Riche-W. in 2019 for $p \geq h$.

Thanks!