Representation theory of monoids

Or: Cell theory for monoids

Daniel Tubbenhauer

Part 2: Reps of algebras; Part 3: Reps of monoidal cats
Where do we want to go?

rep theory
“categorify”
generalize

- Green, Clifford, Munn, Ponizovskii \(\sim 1940++\) + many others
  - Representation theory of (finite) monoids

- Goal: Find some categorical analog
Where do we want to go?

- rep theory
  - “categorify”
  - generalize

- Green, Clifford, Munn, Ponizovskii \( \sim \) 1940++ + many others
  - Representation theory of (finite) monoids

- Goal
  - Find some categorical analog
Where do we want to go?

Talk 1  Monoids and their reps

ON THE STRUCTURE OF SEMIGROUPS

By J. A. Green

(Received June 1, 1950)

Talk 2  The linear version of talk 1

Representations of Coxeter Groups and Hecke Algebras

David Kazhdan¹ and George Lusztig²*

Inventiones math. 53, 165 – 184 (1979)

Talk 3  The categorical version of talk 1

ANALOGUES OF CENTRALIZER SUBALGEBRAS FOR FIAT
2-CATEGORIES AND THEIR 2-REPRESENTATIONS

MARCO MACKAAY¹,², VOLODYMYR MAZORCHUK³, VANESSA MIEMIETZ¹
AND XIAOTING ZHANG⁴,⁵

(Received 23 February 2018; revised 5 November 2018; accepted 7 November 2018;
first published online 4 December 2018)
The theory of monoids (Green ~1950++)

- Associativity $\Rightarrow$ reasonable theory of matrix reps
- Southeast corner $\Rightarrow$ reasonable theory of matrix reps
Adjoining identities is “free” and there is no essential difference between semigroups and monoids.

The main difference is **monoids vs. groups**.

I will stick with the more familiar monoids and groups.

- **Associativity** ⇒ reasonable theory of matrix reps
- **Southeast corner** ⇒ reasonable theory of matrix reps

Examples:

- Groups
- Multiplicative closed sets of matrices (these need not to be unital, but anyway)
- Symmetric groups
- \( Aut(\{1, \ldots, n\}) \)
- Transformation monoids
  - \( End(\{1, \ldots, n\}) \)

In a monoid information is destroyed.

The point of monoid theory is to keep track of information loss.

Monoids appear naturally in categorification.

Examples:

- \( \mathbb{Z} \) is a group
- \( \mathbb{N} \) is a monoid
- \( \mathbb{C}_n = \langle a | a^n = 1 \rangle \) is a group
- \( \mathbb{C}_n, p = \langle a | a^n + p = a^n \rangle \) is a monoid
- \( S_n = Aut(\{1, \ldots, n\}) \) is a group
- \( T_n = End(\{1, \ldots, n\}) \) is a monoid

Finite groups are kind of random... Monoids have almost no structure and there are zillions of them.

⇒ not clear that there is a satisfying (rep) theory of monoids.

**Spoiler** There is ;-)

Example (group-like):
All invertible elements form the smallest cell.

Example (cells of \( \mathbb{N} \)):
Every element is in its own cell, only 0 is idempotent.

Example (cells of \( \mathbb{C}_3, 2 \), idempotent cells colored):

Example (cells of \( T_3 \), idempotent cells colored; more in a second):

Computing these “egg box diagrams” is one of the main tasks of monoid theory.

GAP can do these calculations for you (package semigroups).

Examples (no specific monoids):

- Grey boxes are idempotent
- \( H \)-cells

**Cell theory for algebras**

**Representation theory of monoids**

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<table>
<thead>
<tr>
<th>Group-like structures</th>
<th>Totality</th>
<th>Associativity</th>
<th>Identity</th>
<th>Invertibility</th>
<th>Commutativity</th>
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<td>Commutative monoid</td>
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<td>Unneeded</td>
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<tr>
<td>Group</td>
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<td>Required</td>
</tr>
</tbody>
</table>
The theory of monoids (Green \(\sim 1950++\))

Examples of monoids

Groups

Multiplicative closed sets of matrices (these need not to be unital, but anyway)

Symmetric groups \(\text{Aut}(\{1, \ldots, n\})\)

Transformation monoids \(\text{End}(\{1, \ldots, n\})\)

Groups

Southeast corner \(\Rightarrow\) reasonable theory of matrix reps
The theory of monoids (Green \(\sim 1950++\))

- **Example**
  - \(\mathbb{Z}\) is a group
  - \(\mathbb{N}\) is a monoid

- **Example**
  - \(C_n = \langle a \mid a^n = 1 \rangle\) is a group
  - \(C_{n,p} = \langle a \mid a^{n+p} = a^n \rangle\) is a monoid

- **Example (now with notation)**
  - \(S_n = \text{Aut}(\{1, \ldots, n\})\) is a group
  - \(T_n = \text{End}(\{1, \ldots, n\})\) is a monoid

- Associativity \(\Rightarrow\) reasonable theory of matrix reps
- Southeast corner \(\Rightarrow\) reasonable theory of matrix reps

- Adjoining identities is "free" and there is no essential difference between semigroups and monoids
- The main difference is monoids vs. groups
- I will stick with the more familiar monoids and groups
- In a monoid information is destroyed
- The point of monoid theory is to keep track of information loss
- Monoids appear naturally in categorification
- Examples of monoids:
  - Groups
  - Multiplicative closed sets of matrices (these need not be unital, but anyway)
  - Symmetric groups
  - \(\text{Aut}(\{1, \ldots, n\})\)
  - Transformation monoids
  - \(\text{End}(\{1, \ldots, n\})\)

- Finite groups are kind of random...
- Monoids have almost no structure and there are zillions of them
  - \(\Rightarrow\) not clear that there is a satisfying (rep) theory of monoids
- **Spoiler** There is ;-)
The theory of monoids (∼1950+)

▶ Associativity ⇒ reasonable theory of matrix reps

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Example (group-like):

- All invertible elements form the smallest cell

Example (cells of \(\mathbb{N}\)):

- Every element is in its own cell, only 0 is idempotent

Example (cells of \(C_3, 2\), idempotent cells colored)

Examples (no specific monoids):

- Grey boxes are idempotent, \(H\)-cells

Cell theory for algebras

Representation theory of monoids

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The theory of monoids (∼1950+)

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- Multiplicative closed sets of matrices (these need not to be unital, but anyway)
- Symmetric groups
- Aut(\{1, \ldots, n\})
- Transformation monoids
- End(\{1, \ldots, n\})

Example: \(\mathbb{Z}\) is a group, \(\mathbb{N}\) is a monoid.

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The theory of monoids (Green \(\sim 1950\++\))

The cell orders and equivalences:

\[
\begin{align*}
x \leq_L y &\iff \exists z : y = zx \\
x \leq_R y &\iff \exists z' : y = xz' \\
x \leq_{LR} y &\iff \exists z, z' : y = zxz'
\end{align*}
\[
\begin{align*}
x \sim_L y &\iff (x \leq_L y) \land (y \leq_L x) \\
x \sim_R y &\iff (x \leq_R y) \land (y \leq_R x) \\
x \sim_{LR} y &\iff (x \leq_{LR} y) \land (y \leq_{LR} x)
\end{align*}
\]

Left, right and two-sided cells (a.k.a. \(L, R\) and \(J\)-cells): equivalence classes

- **\(H\)-cells** = intersections of left and right cells
- **Slogan** Cells measure information loss
The theory of monoids (Green ∼1950++)

- Cells partition monoids into matrix-type-pieces
- $L$ and $R$-cells ↔ columns/rows

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The point of monoid theory is to keep track of information loss. Monoids appear naturally in categorification.

Examples of monoids:
- Groups
- Multiplicative closed sets of matrices (these need not to be unital, but anyway)
- Symmetric groups $\text{Aut}(\{1, \ldots, n\})$
- Transformation monoids $\text{End}(\{1, \ldots, n\})$

Examples (now with notation):
- $\mathbb{Z}$ is a group (Integers)
- $\mathbb{N}$ is a monoid (Natural numbers)
- $C_n = \langle a \mid a^n = 1 \rangle$ is a group (Cyclic group)
- $C_{n, p} = \langle a \mid a^{n+p} = a^n \rangle$ is a monoid (Cyclic monoid)
- $S_n = \text{Aut}(\{1, \ldots, n\})$ is a group (Symmetric group)
- $T_n = \text{End}(\{1, \ldots, n\})$ is a monoid (Transformation monoid)

Finite groups are kind of random... Monoids have almost no structure and there are zillions of them. ⇒ not clear that there is a satisfying (rep) theory of monoids. Spoiler: There is ;-) Computing these "egg box diagrams" is one of the main tasks of monoid theory. GAP can do these calculations for you (package semigroups).
The theory of monoids (Green ~1950++)

- $I$-cells = intersections of left and right cells
- The $J$-cells are matrices with values in $H$-cells

$H$-cells = intersections of left and right cells

$H\left( L, R \right)$

$H_1^{11}$, $H_1^{12}$, $H_1^{13}$, $H_1^{14}$, $H_2^{21}$, $H_2^{22}$, $H_2^{23}$, $H_2^{24}$, $H_3^{31}$, $H_3^{32}$, $H_3^{33}$, $H_3^{34}$
The theory of monoids (Green ~1950++)

Each $H$ contains no or 1 idempotent $e$; every $e$ is contained in some $H(e)$.

Each $H(e)$ is a maximal subgroup. No internal information loss.
The theory of monoids (Green ∼1950+)

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<table>
<thead>
<tr>
<th>$H$</th>
<th>$H(e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{11}$</td>
<td>All invertible elements form the smallest cell</td>
</tr>
<tr>
<td>$H_{21}$</td>
<td></td>
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<tr>
<td>$H_{22}$</td>
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<td>$H_{23}$</td>
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<td>$H_{24}$</td>
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<td>$H_{31}$</td>
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<td>$H_{32}$</td>
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<tr>
<td>$H_{33}$</td>
<td></td>
</tr>
<tr>
<td>$H_{34}$</td>
<td></td>
</tr>
</tbody>
</table>

Example (group-like)

All invertible elements form the smallest cell.

Example (cells of $N$)

Every element is in its own cell, only 0 is idempotent.

Example (cells of $C_3$, $P$, idempotent cells colored)

Example (cells of $T_3$, idempotent cells colored; more in a second)

Computing these “egg box diagrams” is one of the main tasks of monoid theory.

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Examples (no specific monoids)

Grey boxes are idempotent $H$-cells.

Cell theory for algebras

Representation theory of monoids

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- Each $\mathcal{H}$ contains no or 1 idempotent $e$; every $e$ is contained in some $\mathcal{H}(e)$
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No internal information loss

Example (cells of \(\mathbb{N}\))

Every element is in its own cell, only 0 is idempotent

Example (cells of \(C_{3,2}\), idempotent cells colored)

\[
\begin{align*}
\mathcal{J}_t & : a^3, a^4 & \mathcal{H}(e) \cong \mathbb{Z}/2\mathbb{Z} \\
\mathcal{J}_{a^2} & : a^2 \\
\mathcal{J}_a & : a \\
\mathcal{J}_b & : 1 & \mathcal{H}(e) \cong 1
\end{align*}
\]

Example (cells of \(T_{3}\), idempotent cells colored; more in a second)

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\(\mathcal{H}_{-}\)-cells

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\mathcal{I}_a & a \\
\mathcal{I}_b & 1 & \mathcal{H}(e) \cong 1
\end{array}
\]

Example (cells of $T_3$, idempotent cells colored; more in a second)

Each $\mathcal{H}(e)$ is a maximal subgroup

$\mathcal{H}(e)$

Cell theory for algebras

Representation theory of monoids
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Examples (no specific monoids):

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Example (cells of $C_3 \times C_2$, idempotent cells colored):

Example (cells of $T_3$, idempotent cells colored; more in a second).

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Cell theory for algebras

Representation theory of monoids

August 2022
The theory of monoids (Green \(\sim 1950++\))

Each \(H\) contains no or 1 idempotent \(e\); every \(e\) is contained in some \(\mathcal{H}(e)\).

- Each \(\mathcal{H}(e)\) is a maximal subgroup.
- No internal information loss.

Examples (no specific monoids)

Grey boxes are idempotent \(H\)-cells.

Examples (no specific monoids)

Cell theory for algebras
Representation theory of monoids
Cells of some diagram monoids

Connect eight points at the bottom with eight points at the top:

(24138567) ↔

or

(24637158) ↔

We just invented the symmetric group $S_8$ on \{1, ..., 8\}
Cells of some diagram monoids

My multiplication rule for $gh$ is “stack $g$ on top of $h$”
Cells of some diagram monoids

- We clearly have $g(hf) = (gh)f$
- There is a do nothing operation $1g = g = g1$

Generators–relations (the Reidemeister moves), e.g.

gens : , rels : = , =
Cells of some diagram monoids

Allow merges and top dots:

(23135555) \leftrightarrow top dot
merge or

(11335577) \leftrightarrow

We just invented the transformation monoid $T_8$ on $\{1, \ldots, 8\}$
My multiplication rule for $gh$ is “stack $g$ on top of $h$”
Cells of some diagram monoids

- Generators–relations for $S_n \subset T_n$ (the Reidemeister moves), e.g.
  
  \[
  \text{gens : } \quad \quad \text{rels : } \quad \quad = \quad \quad =
  \]

- Generators–relations for the non-invertible part of $T_n$, e.g.
  
  \[
  \text{gens : } \quad \quad \text{rels : } \quad \quad = \quad \quad =
  \]

- Interactions, e.g.
  
  \[
  = \quad \quad =
  \]
Cells of some diagram monoids

Generators–relations for $S_n \subset T_n$ (the Reidemeister moves), e.g.

- **Gens:** $\epsilon$, $\sigma$, $\tau$
- **Rel:** $\epsilon \circ \sigma = \sigma \circ \epsilon$

Generators–relations for the non-invertible part of $T_n$, e.g.

- **Gens:** $\epsilon$, $\sigma$, $\tau$
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Interactions, e.g.

- **Gens:** $\epsilon$, $\sigma$, $\tau$
- **Rel:** $\epsilon \circ \sigma = \sigma \circ \epsilon$

Theorem (folklore)

- $J$-cells of $T_n$ are given and ordered by strands.
- All $J$-cells contain idempotents.
- $L$-cells correspond to fixed bottom ($\{n\}_{\lambda}$ many), $R$-cells to fixed top ($\{(n\lambda)\}_{\lambda}$ many).

Example (cells of $T_3$, idempotent cells colored)

- $J_t$
  - (111)
  - (222)
  - (333)
  - $\mathcal{H}(e) \cong S_1$

- $J_m$
  - (122), (211)
  - (121), (212)
  - (221), (112)
  - (133), (311)
  - (313), (131)
  - (113), (331)
  - (233), (322)
  - (323), (232)
  - (223), (332)
  - $\mathcal{H}(e) \cong S_2$

- $J_b$
  - (123), (213), (132)
  - (231), (312), (321)
  - $\mathcal{H}(e) \cong S_3$

Examples (cell theory for algebras, representation theory of monoids)

- Planar partition monoid $\text{ppA}_4$, $\text{ppA}_4$ via GAP
- Motzkin + rook Brauer monoid $\text{Mo}_4$, $\text{RoBr}_4$ via GAP
- Temperley–Lieb + Brauer monoid $\text{TL}_4$, $\text{Br}_4$ via GAP
- Planar rook monoid $\text{pRo}_3$, $\text{Ro}_3$ by hand

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Cells of some diagram monoids

Generators–relations for $S_n \subset T_n$ (the Reidemeister moves), e.g.

<table>
<thead>
<tr>
<th>$\mathcal{I}_t$</th>
<th>$\mathcal{I}_m$</th>
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<td>(111)</td>
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$\mathcal{H}(e) \cong S_1$

$\mathcal{H}(e) \cong S_2$

$\mathcal{H}(e) \cong S_3$

Example (cells of $T_3$, idempotent cells colored)

$\text{J}_t$

$\text{J}_m$

$\text{J}_b$

Interactions, e.g.

Example (cells of $T_3$, idempotent cells colored)

$\text{J}_t$

$\text{J}_m$

$\text{J}_b$

$\mathcal{H}(e) \cong S_1$

$\mathcal{H}(e) \cong S_2$

$\mathcal{H}(e) \cong S_3$
Cells of some diagram monoids

Theorem (folklore)

J-cells of $T_n$ are given and ordered by through strands $\lambda$

All J-cells contain idempotents

$L$-cells correspond to fixed bottom ($\{\binom{n}{\lambda}\}$ many), $R$-cells to fixed top ($\binom{n}{\lambda}$ many)

$\mathcal{H}(e) \cong S_\lambda$ for $\lambda = \#$ through strands

Example (cells of $T_3$, idempotent cells colored)

Generators–relations for the non-invertible part of $T_n$, e.g.

$\mathcal{H}(e) \cong S_1$

$\mathcal{H}(e) \cong S_2$

$\mathcal{H}(e) \cong S_3$

Interactions, e.g.
Cells of some diagram monoids

- Generators–relations for $S_n \subset T_n$ (the Reidemeister moves), e.g.
  - gens: $\times$, $\cdot$

- Generators–relations for the non-invertible part of $T_n$, e.g.
  - gens: $\nabla$, $\wedge$

- Interactions, e.g.
  - $\psi = \sigma$

Example ($T_5$ via GAP)

Examples ((planar) partition monoid $pPa_4$ via GAP)

Examples (Motzkin + rook Brauer monoid $Mo_4$, $RoBr_4$ via GAP)

Examples (Temperley–Lieb + Brauer monoid $TL_4$, $Br_4$ via GAP)

Examples (planar) rook monoid $pRo_3$, $Ro_3$ by hand

Cell theory for algebras
Representation theory of monoids
August 2022
Cells of some diagram monoids

More examples (details on the exercise sheets)
Planar (left) and symmetric (right) diagram monoids, e.g.

<table>
<thead>
<tr>
<th>Symbol</th>
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<tr>
<td>pPa_n</td>
<td><img src="pPa.png" alt="Diagram" /></td>
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The (planar) symmetric groups pS\_n, S\_n are groups ⇒ Boring cells
Cells of some diagram monoids

More examples (details on the exercise sheets)

Planar (left) and symmetric (right) diagram monoids, e.g.

The (planar) symmetric groups $\mathfrak{S}_n, \mathfrak{S}_n$ are groups ⇒ Boring cells

Example (cells of $\mathcal{T}_3$)

Example (cells of $\mathcal{T}_3$)

Theorem (folklore)

$J$-cells of $\mathcal{T}_n$ are given and ordered by

All $J$-cells contain idempotents

$L$-cells correspond to fixed bottom ($\{n_\lambda\}$ many),

$R$-cells to fixed top ($\{n\}$ many)

$H(e) \sim = \mathfrak{S}_\lambda$ for $\lambda = \# through strands

Example ($\mathcal{T}_5$ via GAP)

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Cell theory for algebras

Representation theory of monoids
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Planar (left) and symmetric (right) diagram monoids, e.g.

The (planar) symmetric groups $pS_n$, $S_n$ are groups $\Rightarrow$

Examples (cells of $T_3$, idempotent cells colored)

Theorem (folklore)

$J$-cells of $T_n$ are given and ordered by through strands

All $J$-cells contain idempotents

$L$-cells correspond to fixed bottom (many),

$R$-cells to fixed top (many)

$H(e) \sim = S_\lambda$ for $\lambda = # through strands

Examples (via GAP)

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Cell theory for algebras

Representation theory of monoids

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Cells of some diagram monoids

More examples (details on the exercise sheets)

Examples ((planar) rook monoid $pR_{o3}, R_{o3}$ by hand)

The (planar) symmetric groups $pS_n, S_n$ are groups ⇒ Boring cells
The simple reps of monoids

\[ \phi: S \to \text{GL}(V) \]  
\( S \)-representation on a \( \mathbb{K} \)-vector space \( V \), \( S \) is some monoid

- A \( \mathbb{K} \)-linear subspace \( W \subset V \) is \( S \)-invariant if \( S \cdot W \subset W \) \( \text{Substructure} \)
- \( V \neq 0 \) is called simple if \( 0, V \) are the only \( S \)-invariant subspaces \( \text{Elements} \)
- Careful with different names in the literature: \( S \)-invariant \( \iff \) subrepresentation, simple \( \iff \) irreducible
- A crucial goal of representation theory

Find the periodic table of simple \( S \)-representations

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The simple reps of monoids

φ: S → GL(V) S-representation on a K-vector space V, S is some monoid

- A K-linear subspace W ⊂ V is S-invariant if S W ⊂ W
- V ≠ 0 is called simple if 0, V are the only S-invariant subspaces
- Careful with different names in the literature: S-invariant ↔ subrepresentation, simple ↔ irreducible
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\[ \phi: S \to \text{GL}(V) \] is an \( S \)-representation on a \( K \)-vector space \( V \), \( S \) is some monoid.

- A \( K \)-linear subspace \( W \subset V \) is \( S \)-invariant if \( S \cdot W \subset W \).
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Careful: \( S \)-invariant \( \iff \) subrepresentation, simple \( \iff \) irreducible.

A crucial goal of representation theory: Find the periodic table of simple \( S \)-representations.

- Frobenius \( \sim 1895++ \) and others
- For groups and \( K = \mathbb{C} \) rep theory is really satisfying.

What about monoids?

Me: Probably not much better than general algebra rep theory...
Jeez, was I wrong...

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Substructure

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Elements

- Careful with different names in the literature: \( S \)-invariant \( \leftrightarrow \) subrepresentation, simple \( \leftrightarrow \) irreducible

A crucial goal of representation theory

Find the periodic table of simple \( S \)-representations

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*MATRIX REPRESENTATIONS OF COMPLETELY SIMPLE SEMIGROUPS.* 1942

By A. H. Clifford.

ON SEMIGROUP ALGEBRAS

By W. D. Munn

Received 21 July 1954

О матричных представлениях ассоциативных систем* 1956

И. С. Понизовский (Кемерово)

The rep theory of monoids is much better than expected!
The simple reps of monoids

Clifford, Munn, Ponizovskii \( \sim 1940 \) \( \uparrow \) \( H \)-reduction

There is a one-to-one correspondence

\[
\begin{align*}
\left\{ \text{simples with apex } J(e) \right\} & \overset{\text{one-to-one}}{\leftrightarrow} \left\{ \text{simples of (any) } H(e) \subset J(e) \right\} \\
\end{align*}
\]

Reps of monoids are controlled by \( H(e) \)-cells

- Each simple has a unique maximal \( J(e) \) whose \( H(e) \) does not kill it \( \text{Apex} \)
- In other words (smod means the category of simples):

\[
S\text{-smod}_{J(e)} \cong H(e)\text{-smod}
\]
There is a one-to-one correspondence\n\[
\{ \text{simpleres of (any)} \ H(e) \subset J(e) \}\n\longleftrightarrow \\{ \text{simpleres of } J(e) \}\n\]

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Example (groups)

Groups have only one cell – the group itself

\( H(e) \)-reduction is trivial for groups

Example (cells of \( C_3 \times 2 \), idempotent cells colored)

Three simple reps over \( C_3 \times 2 \):

one for \( J_b \) and two for \( J_t \)

Example (cells of \( T_3 \), idempotent cells colored)

Six simple reps over \( C_3 \times 2 \):

three for \( J_b \), two for \( J_m \) and one for \( J_t \)

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Trivial rep of \( 1 \) induces to \( C_3 \times 2 \) and has apex \( J_b \)

Trivial rep of \( \mathbb{Z}/2\mathbb{Z} \) induces to \( C_3 \times 2 \) and has apex \( J_t \)

Nothing acts by zero

Example (no specific monoid)

Five apexes: bottom cell, big cell, 2x2 cell, 3x3 cell, top cell

Simples for the 2x2 cell are acted on as zero by elements from 3x3 cell, top cell

\( H(e) \)-reduction

It is sufficient to pick one \( H(e) \) per block

\( J \)-reduction = existence of apexes

Basically, there is a monoid \( S_J \) associated to fish with

Simples of \( S_J \) \( \xrightarrow{1:1} \) simples of \( S \) with apex fish

“Apex = fish” means that the red bubble does not annihilate your rep and the rest does
The simple reps of monoids

Clifford, Munn, Ponizovskiǐ \(\sim 1940\)++ 

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The simple reps of monoids
Clifford, Munn, Ponizovski
\sim 1940

There is a one-to-one correspondence
\{ simples \} \leftrightarrow \{ simples of (any) \}
\mathcal{J}(e) \subset \mathcal{J}(e)

Reps of monoids are controlled by \mathcal{H}(e)-cells

- Each simple has a unique maximal \mathcal{J}(e) whose \mathcal{H}(e) does not kill it
- In other words (smod means the category of simples):

$$S\text{-smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-smod}$$
The simple reps of monoids

Clifford, Munn, Ponizovski \( \sim \) 1940+

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Frobenius \( \sim \) 1895+ and others

For groups and \( K = C \) rep theory is really satisfying

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Example (anti apex predator)

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\[
\text{Simples of } S \cong \text{Simples of } S \text{ with apex fish}
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Example (cells of \( T_3 \), idempotent cells colored)

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- three for \( J_b \)
- two for \( J_t \)
- one for \( J_t \)

Trivial rep of \( 1 \) induces to \( C_3, 2 \) and has apex \( J_b \)

\( J_a, J_{a^2}, J_t \) act by zero

Trivial rep of \( \mathbb{Z}/2\mathbb{Z} \) induces to \( C_3, 2 \) and has apex \( J_t \)

Nothing acts by zero

Cell theory for algebras

Representation theory of monoids

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There is a one-to-one correspondence:
\[
\{\text{simples with apex } J(e) \} \leftrightarrow \{\text{simples of } (H \cap e) \subset J(e) \}\]

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Three simple reps over \(C_3\):
one for \(J_b\) and two for \(J_t\).

Example (cells of \(T_3\), idempotent cells colored)
Six simple reps over \(C_3\):
three for \(J_b\), two for \(J_m\) and one for \(J_t\).

Trivial rep of \(1\) induces to \(C_3, 2\) and has apex \(J_b\), \(J_a\), \(J_a^2\), \(J_t\) act by zero.
Trivial rep of \(\mathbb{Z}/2\mathbb{Z}\) induces to \(C_3, 2\) and has apex \(J_t\), nothing acts by zero.

Example (no specific monoid)
Five apexes: bottom cell, big cell, 2x2 cell, 3x3 cell, top cell.
Simples for the 2x2 cell are acted on as zero by elements from 3x3 cell, top cell.

\(H\)-reduction It is sufficient to pick one \(H(e)\) per block.
The simple reps of monoids

Clifford, Munn, Ponizovskii ~1940++ \(H\)-reduction

► There are **cell representations**

Cells can be considered S-representations, called cell representations or Schützenberger representations, up to higher order terms:

**Lemma 3B.1.** Each left cell \(L\) of \(S\) gives rise to a left \(S\)-representation \(\Delta_L = \mathbb{K}L\) by

\[
\delta \cdot l \in \Delta_L = \begin{cases} 
al & \text{if } a \in L, \\
0 & \text{else.}
\end{cases}
\]

Similarly, right cells give right representations \(\Delta_R\) and \(J\)-cells give birepresentations (often called bimodules). We have \(\dim_{\mathbb{K}}(\Delta_L) = |L|\) and \(\dim_{\mathbb{K}}(\Delta_R) = |R|\).

► There is a sandwich matrix which takes values in the \(H\)-cells

► There is an **isomorphism of rings**

\[
[S\text{-mod}] \cong \prod_{J(e)} [H(e)\text{-mod}]
\]

► \(S\) is semisimple if and only if all \(J\)-cells are idempotent and square, all \(H(e)\) are semisimple + a condition on cell representations

► Many more...
The simple reps of some diagram monoids

The transformation monoid $T_3$ has three apexes, five left cell modules $\Delta(\lambda, i)$, seven right cell modules $\nabla(\lambda, i)$.

Over $\mathbb{C}$ we find $3+2+1$ simple modules.
The simple reps of some diagram monoids

The transformation monoid $T_3$ has three apexes, five left cell modules $\Delta(\lambda, i)$, seven right cell modules $\nabla(\lambda, i)$.

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Summary: H-reduction reduces monoid rep theory to group rep theory.
The simple reps of some diagram monoids

The transformation monoid \( T_3 \) has three apexes, five left cell modules \( \Delta(\lambda, i) \), seven right cell modules \( \nabla(\lambda, i) \).

Over \( \mathbb{C} \) we find 3 + 2 + 1 simple modules.

The bottom cell

\( \Delta(b) \) is the regular rep of \( S_3 \) inflated to \( T_3 \):

\[
\begin{align*}
\Delta(b) & \in J_b \quad \text{but} \quad \Delta(b) \notin J_b
\end{align*}
\]

The middle cell, left column (the others are similar)

\( \Delta(m, 1) \) is the regular rep of \( S_2 \) induced to \( T_3 \):

\[
\begin{align*}
\Delta(m, 1) & \in J_b \quad \text{and} \quad \Delta(m, 1) \in J_b
\end{align*}
\]

\[
\begin{align*}
\Delta(m, 1) & \notin J_b
\end{align*}
\]

The top cell

\( \Delta(t) \) is the regular rep of \( S_1 \) induced to \( T_3 \)

Theorem (folklore)

The simple \( T_n \)-reps are \( L(\lambda, K) \) for \( K \) a simple \( S_\lambda \)-rep.

Unless \( K \) is the sign \( S_\lambda \)-rep the induction to the cell is simple.

For \( K = \text{sign} \) the \( L(\lambda, K) \) are of dimension \((n - 1)\lambda - 1)\).

Summary

H-reduction reduces monoid rep theory to group rep theory.
The simple reps of some diagram monoids

**The bottom cell over \( \mathbb{C} \)**

\( \Delta(b) \) contributes three simple of \( S_3 \) that inflate to \( T_3 \)
dims are 1, 2, 1 as \( T_3 \) reps

**The transformation monoid** \( T_3 \) has three apexes, five left cell modules \( \Delta(\lambda, i) \),
seven right cell modules \( \nabla(\lambda, i) \)

**Over** \( \mathbb{C} \) we find **3+2+1** simple modules

**Theorem (folklore)**
The simple \( T_n \)-reps are \( L(\lambda, K) \) for \( K \) a simple \( S_\lambda \)-rep
Unless \( K \) is the sign \( S_\lambda \)-rep the induction to the cell is simple
For \( K = \text{sign} \) the \( L(\lambda, K) \) are of dimension \( (n-1)\lambda - 1 \)

**Summary**
\( H \)-reduction reduces monoid rep theory to group rep theory
The simple reps of some diagram monoids

The bottom cell over $\mathbb{C}$

$\Delta(b)$ contributes three simple of $S_3$ that inflate to $T_3$

dims are 1, 2, 1 as $T_3$ reps

$\mathcal{H}(e) \cong S_1$

The middle cell over $\mathbb{C}$

$\Delta(b, 1)$ contributes two simple of $S_2$ that induce to $T_3$ (one of them decomposes), e.g.

is an $S_2$-invariant vector + track its image $\leadsto$ simple
dims are 3, 2 as $T_3$ reps

- The transformation monoid $T_3$ has three apexes, five left cell modules $\Delta(\lambda, i)$, seven right cell modules $\nabla(\lambda, i)$
- Over $\mathbb{C}$ we find $3+2+1$ simple modules
The simple reps of some diagram monoids

The bottom cell over $\mathbb{C}$

$\Delta(b)$ contributes three simple of $S_3$ that inflate to $T_3$
dims are 1, 2, 1 as $T_3$ reps

The middle cell over $\mathbb{C}$

$\Delta(b, 1)$ contributes two simple of $S_2$ that induce to $T_3$ (one of them decomposes), e.g.

\[
+ \xrightarrow{\sim} \text{simple}
\]

dims are 3, 2 as $T_3$ reps

The top cell

$\Delta(t)$ contributes the trivial $T_3$ module
dim is 1 as $T_3$ rep
The transformation monoid $T_3$ has three apexes, five left cell modules $\Delta(\lambda, i)$, seven right cell modules $\nabla(\lambda, i)$.

Over $\mathbb{C}$ we find $3+2+1$ simple modules.
The simple reps of some diagram monoids

The transformation monoid $T_3$ has three apexes, five left cell modules $\Delta(\lambda, i)$, seven right cell modules $\nabla(\lambda, i)$.

Over $\mathbb{C}$ we find $3+2+1$ simple modules.

Sandwich matrices for the middle cell

$S_{m, \text{triv}} = \begin{pmatrix}
\text{triv} & \text{triv} & \text{triv} & \text{triv} & 0 & 0 \\
\text{triv} & \text{triv} & \text{triv} & \text{triv} & 0 & 0 \\
\text{triv} & \text{triv} & 0 & 0 & \text{triv} & \text{triv} \\
\text{triv} & \text{triv} & 0 & 0 & \text{triv} & \text{triv} \\
0 & 0 & \text{triv} & \text{triv} & \text{triv} & \text{triv} \\
0 & 0 & \text{triv} & \text{triv} & \text{triv} & \text{triv}
\end{pmatrix}$

$S_{m, \text{sign}} = \begin{pmatrix}
\text{sign} & -\text{sign} & \text{sign} & -\text{sign} & 0 & 0 \\
-\text{sign} & \text{sign} & -\text{sign} & \text{sign} & 0 & 0 \\
\text{sign} & -\text{sign} & 0 & 0 & -\text{sign} & \text{sign} \\
-\text{sign} & \text{sign} & 0 & 0 & \text{sign} & -\text{sign} \\
0 & 0 & -\text{sign} & \text{sign} & -\text{sign} & \text{sign} \\
0 & 0 & \text{sign} & -\text{sign} & \text{sign} & -\text{sign}
\end{pmatrix}$

Ranks are 3 and 2 = dims of simples

Theorem (folklore)

The simple $T_n$-reps are $L(\lambda, K)$ for $K$ a simple $S_\lambda$-rep

Unless $K$ is the sign $S_\lambda$-rep the induction to the cell is simple

For $K = \text{sign}$ the $L(\lambda, K)$ are of dimension $\binom{n-1}{\lambda-1}$
The simple reps of some diagram monoids

- The Brauer monoid $Br_3$ has two apexes, four left/right cell modules
- Over $\mathbb{C}$ we find $3 + 1$ simple modules
- Other diagram algebras are similar; more on the exercise sheets
The simple reps of some diagram monoids

The Brauer monoid $\text{Br}_3$ has two apexes, four left/right cell modules

Over $\mathbb{C}$ we find $3 + 1$ simple modules

Other diagram algebras are similar; more on the exercise sheets

The bottom cell $\Delta(b)$ is the regular rep of $S_3$ inflated to $T_3$:

$$b \in J \quad \text{but} \quad b \notin J$$

The middle cell, left column (the others are similar)

$\Delta(m, 1)$ is the regular rep of $S_2$ induced to $T_3$:

$$= \in J \quad \text{and} \quad = \notin J$$

The top cell $\Delta(t)$ is the regular rep of $S_1$ induced to $T_3$:

The bottom cell over $\mathbb{C}$ $\Delta(b)$ contributes three simple of $S_3$ that inflate to $T_3$;

dims are 1, 2, 1 as $T_3$ reps

The middle cell over $\mathbb{C}$ $\Delta(b, 1)$ contributes two simple of $S_2$ that induce to $T_3$ (one of them decomposes), e.g.

$+$ is an $S_2$-invariant vector $\Rightarrow$ simple
dims are 3, 2 as $T_3$ reps

The top cell $\Delta(t)$ contributes the trivial $T_3$ module

dim is 1 as $T_3$ rep

Sandwich matrices for the middle cell

Ranks are 3 and 2 = dims of simples

Theorem (folklore)

The simple $T_n$-reps are $L(\lambda, K)$ for $K$ a simple $S_\lambda$-rep

Unless $K$ is the sign $S_\lambda$-rep the induction to the cell is simple

For $K = \text{sign}$ the $L(\lambda, K)$ are of dimension

$$\left( n - 1 \right)$$

Summary

$H$-reduction reduces monoid rep theory to group rep theory

Clifford, Munn, Ponizovskii $\sim 1940$++ (H-reduction)

There is a one-to-one correspondence

$$\left\{ \text{simples with apex } \mathcal{J}(e) \right\} \xleftrightarrow{\text{one-to-one}} \left\{ \text{simples of (any) } \mathcal{H}(e) \subseteq \mathcal{J}(e) \right\}$$

Reps of monoids are controlled by $\mathcal{H}(e)$ cells
There is still much to do...
Where do we want to go?

- Green, Clifford, Munn, Ponizovskii –1940++ + many others
- Goal: Find some categorical analog
- Appear in interactions, e.g.
  - The (planar) symmetric groups

The theory of monoids (Green –1950++)

- Monoids appear naturally in categorification
- Cell theory for algebras
- The theory of monoids is not clear that there is a satisfying (rep) theory of monoids
- 

The simple reps of monoids

Clifford, Munn, Ponizovskii –1940++

- There is a one-to-one correspondence
- Simple reps of any $J(e)$:
  \[
  \begin{align*}
  & \text{appl \ } J(e) \\
  \quad \text{one-to-one} \quad & \text{simples of any } J(e) \\
  \quad \text{of any } J(e) \quad & \text{by control by } H(e) \subset J(e)
  \end{align*}
  \]

- Each simple has a unique maximal $J(e)$ whose $H(e)$ does not kill it
- In other words (mod) means the category of simples:
  \[ S\text{-mod}_{\text{mod}} \cong H(e)\text{-mod} \]

The rep theory of monoids is really satisfying

- Finite groups are kind of random...
- Three simple reps over $\mathbb{C}$, idempotent cells colored
- \[ \begin{align*}
  \{ & J_1, J_2, J_3 \} \\
  \{ & J_1, J_2, J_3 \} \\
  \{ & J_1, J_2, J_3 \}
  \end{align*} \]
- Nothing acts by zero

The rep theory of monoids is really satisfying

- Many examples:
  - Temperley-Lieb + Brauer monoid
  - Motzkin + rook Brauer monoid
  - (planar) rook monoid
  - Cyclic group
  - Symmetric group
  - Transformation monoid

Example (anti apex predator)

- Apex $\subseteq$ fish
- Basically, there is a monoid $S_+ \cong J$ associated to fish with simples of $S_+$

Thanks for your attention!