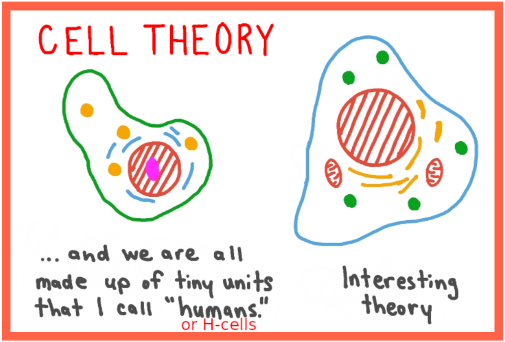
Representation theory of monoids

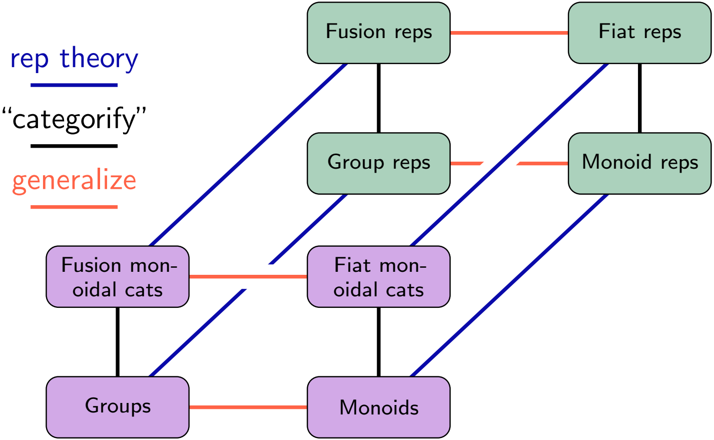
Or: Cell theory for monoids

Daniel Tubbenhauer



Part 2: Reps of algebras; Part 3: Reps of monoidal cats

Where do we want to go?



- **Green, Clifford, Munn, Ponizovskiĭ ∼1940++ + many others**
  Representation theory of (finite) monoids
- Goal Find some categorical analog

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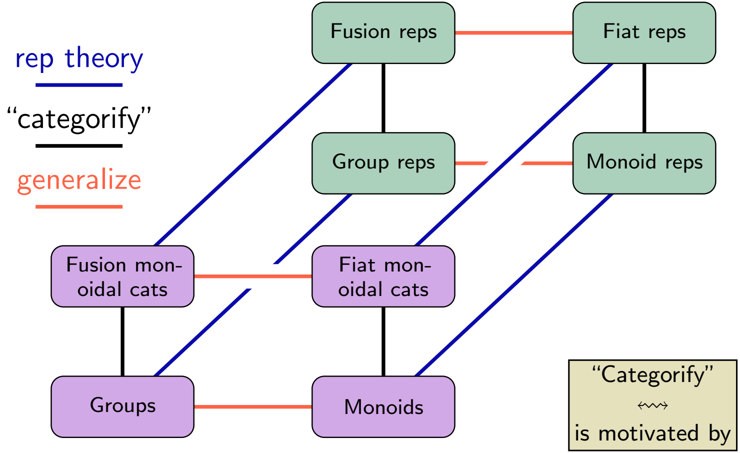

- **Green, Clifford, Munn, Ponizovskiĭ ∼1940++ + many others**
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Where do we want to go?

- **Talk 1** Monoids and their reps

ON THE STRUCTURE OF SEMIGROUPS

By J. A. Green

(Received June 1, 1950)

$$x \leq_L y \Leftrightarrow \exists z: y = zx$$
$$x \leq_R y \Leftrightarrow \exists z': y = xz'$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z': y = zxz'$$

- **Talk 2** The linear version of talk 1

Representations of Coxeter Groups and Hecke Algebras

David Kazhdan[1] and George Lusztig[2]*

Inventiones math. 53, 165–184 (1979)

$$x \leq_L y \Leftrightarrow \exists z: y \in^{+} zx$$
$$x \leq_R y \Leftrightarrow \exists z': y \in^{+} xz'$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z': y \in^{+} zxz'$$

- **Talk 3** The categorical version of talk 1

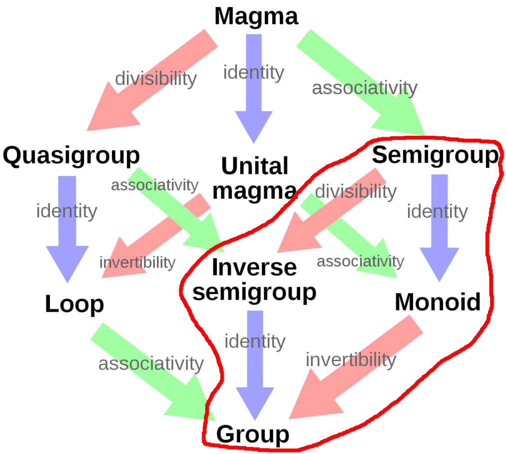
ANALOGUES OF CENTRALIZER SUBALGEBRAS FOR FIAT 2-CATEGORIES AND THEIR 2-REPRESENTATIONS

MARCO MACKAAY[1,2], VOLODYMYR MAZORCHUK[3], VANESSA MIEMIETZ[4] AND XIAOTING ZHANG[5]

(Received 23 February 2018; revised 5 November 2018; accepted 7 November 2018; first published online 4 December 2018)

$$X \leq_L Y \Leftrightarrow \exists Z: Y \in^{\oplus} ZX$$
$$X \leq_R Y \Leftrightarrow \exists Z': Y \in^{\oplus} XZ'$$
$$X \leq_{LR} Y \Leftrightarrow \exists Z, Z': Y \in^{\oplus} ZXZ'$$

The theory of monoids (Green ∼1950++)



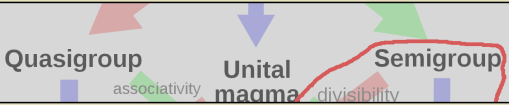
- Associativity ⇒ reasonable theory of matrix reps
- Southeast corner ⇒ reasonable theory of matrix reps

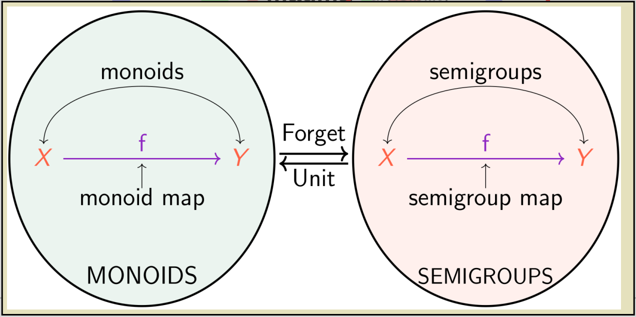
The th

Adjoining identities is "free" and there is no essential difference between semigroups and monoids

The main difference is monoids vs. groups

I will stick with the more familiar monoids and groups





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In a monoid information is destroyed

The point of monoid theory is to keep track of information loss

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I will stick with the more familiar monoids and groups

In a monoid information is destroyed

The point of monoid theory is to keep track of information loss

Monoids appear naturally in categorification

Group-like structures

	Totality[α]	Associativity	Identity	Invertibility	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Inverse semigroup	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Group	Required	Required	Required	Required	Unneeded
Abelian group	Required	Required	Required	Required	Required

- Associativity =
- Southeast corr

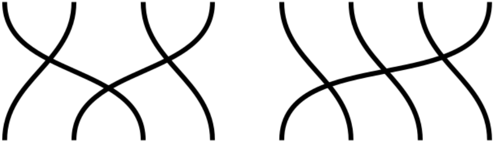
Examples of monoids

Groups

Multiplicative closed sets of matrices (these need not to be unital, but anyway)

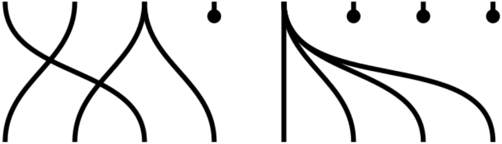
Symmetric groups $\mathrm{Aut}(\{1,\dots,n\})$

$(24138567) \leftrightsquigarrow$



Transformation monoids $\mathrm{End}(\{1,\dots,n\})$

$(23135555) \leftrightsquigarrow$



- Southeast corner $\Rightarrow$ reasonable theory of matrix reps

Example

$\mathbb{Z}$ is a group — Integers

$\mathbb{N}$ is a monoid — Natural numbers

Example

$C_n = \langle a | a^n = 1 \rangle$ is a group — Cyclic group

$C_{n,p} = \langle a | a^{n+p} = a^n \rangle$ is a monoid — Cyclic monoid

Example (now with notation)

$S_n = \mathrm{Aut}(\{1, ..., n\})$ is a group — Symmetric group

$T_n = \mathrm{End}(\{1, ..., n\})$ is a monoid — Transformation monoid

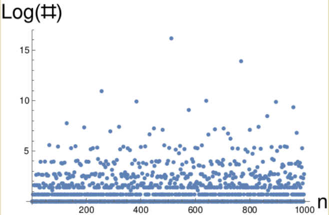
- Associativity ⇒ reasonable theory of matrix reps
- Southeast corner ⇒ reasonable theory of matrix reps

Finite groups are kind of random...

The t

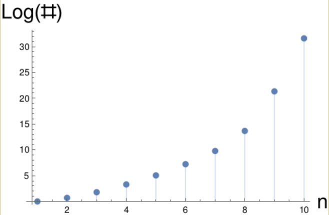
A000001 Number of groups of order n.
(Formerly M0098 N0035)

0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1, 51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1, 2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,



A058133 Number of monoids (semigroups with identity) of order n, considered to be equivalent when they are isomorphic or anti-isomorphic (by reversal of the operator).

0, 1, 2, 6, 27, 156, 1373, 17730, 858977, 1844075697, 52991253973742 (list; graph; refs; listen; history;



▶ A

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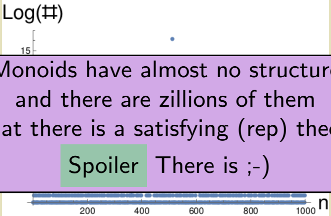
51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1,
2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,



Monoids have almost no structure
and there are zillions of them
⇒ not clear that there is a satisfying (rep) theory of monoids

Spoiler There is ;-)

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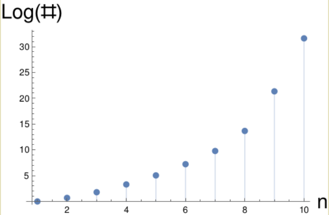
0, 1, 2, 6, 27, 156, 1373, 17730, 858977, 1844075697, 52991253973742 (list; graph; refs; listen; history;



The theory of monoids (Green $\sim$1950++)

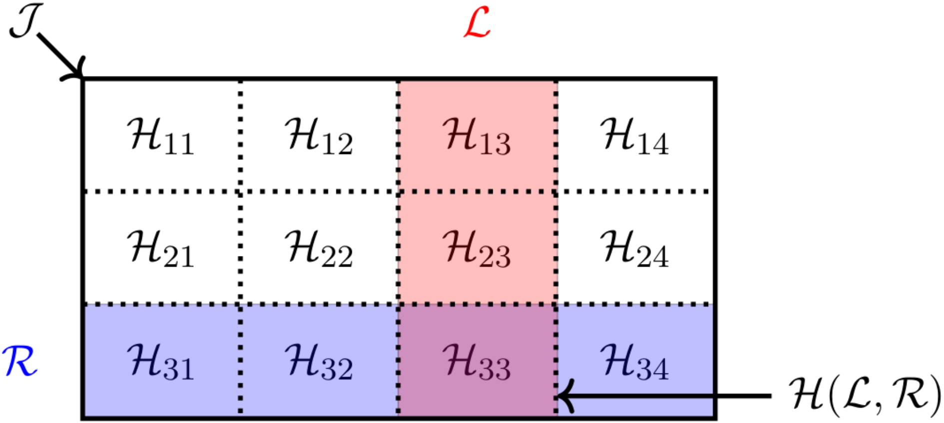
The cell orders and equivalences:

$$x \leq_L y \Leftrightarrow \exists z\colon y = zx$$
$$x \leq_R y \Leftrightarrow \exists z'\colon y = xz'$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z'\colon y = zxz'$$
$$x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x)$$
$$x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x)$$
$$x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x)$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

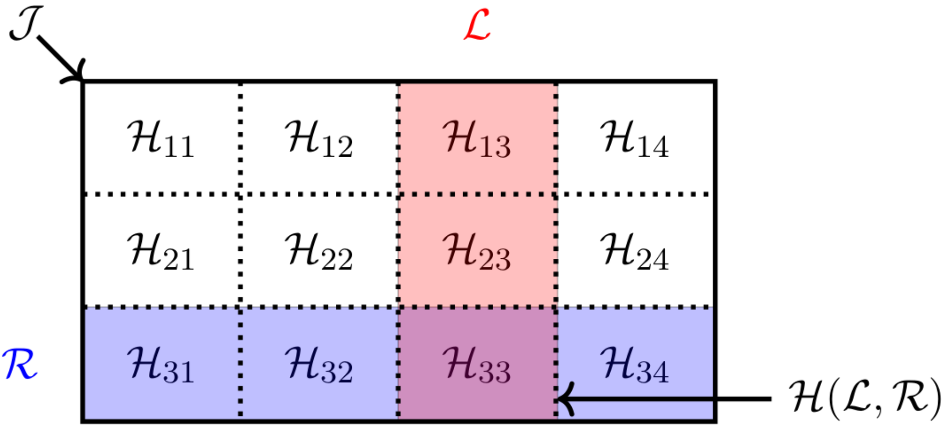
- H-cells = intersections of left and right cells
- Slogan Cells measure information loss

The theory of monoids (Green ∼1950++)



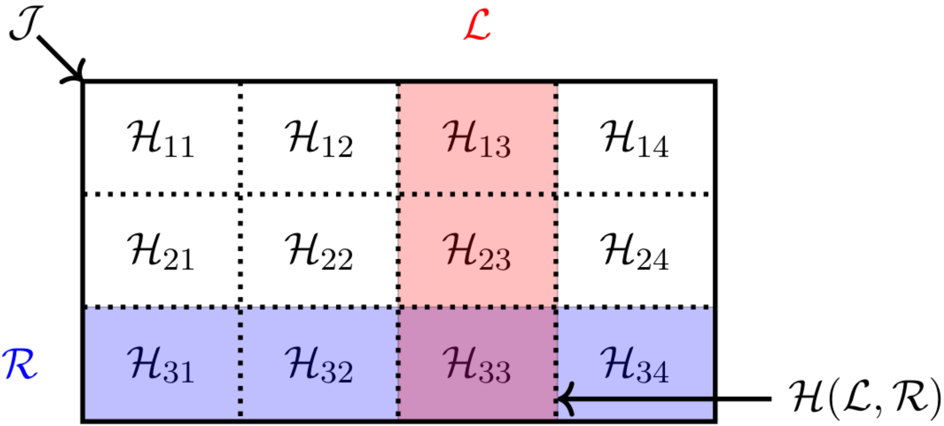
- Cells partition monoids into matrix-type-pieces
- L and R-cells ⇝ columns/rows

The theory of monoids (Green ∼1950++)



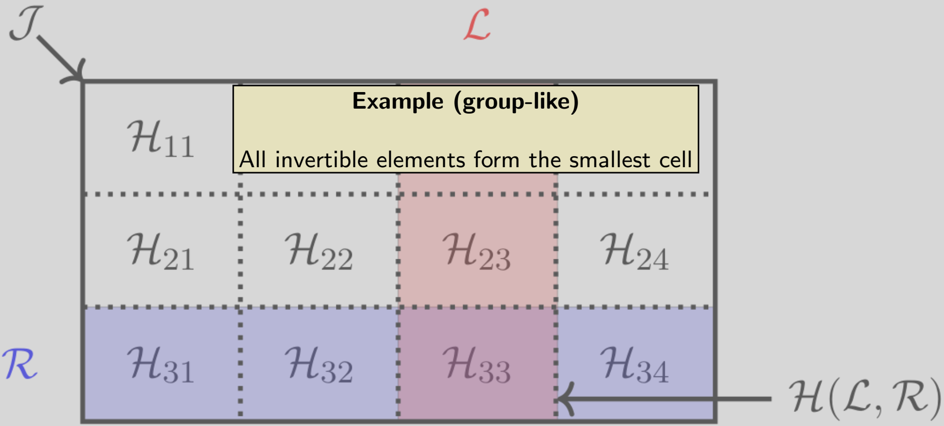
- *H*-cells = intersections of left and right cells
- The *J*-cells are matrices with values in *H*-cells

The theory of monoids (Green ∼1950++)



- Each $\mathcal{H}$ contains no or 1 idempotent e; every e is contained in some $\mathcal{H}(e)$
- Each $\mathcal{H}(e)$ is a maximal subgroup No internal information loss

The theory of monoids (Green ∼1950++)



Example (group-like)

All invertible elements form the smallest cell

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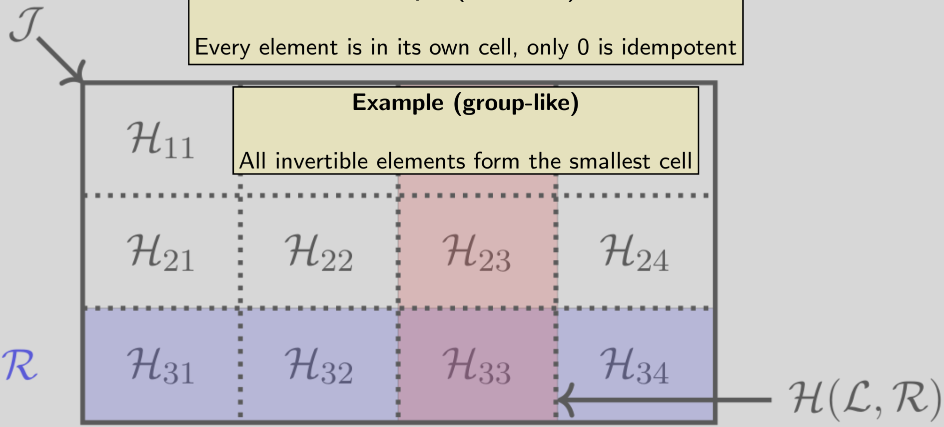
Example (cells of $\mathbb{N}$)

Every element is in its own cell, only 0 is idempotent

Example (group-like)

All invertible elements form the smallest cell



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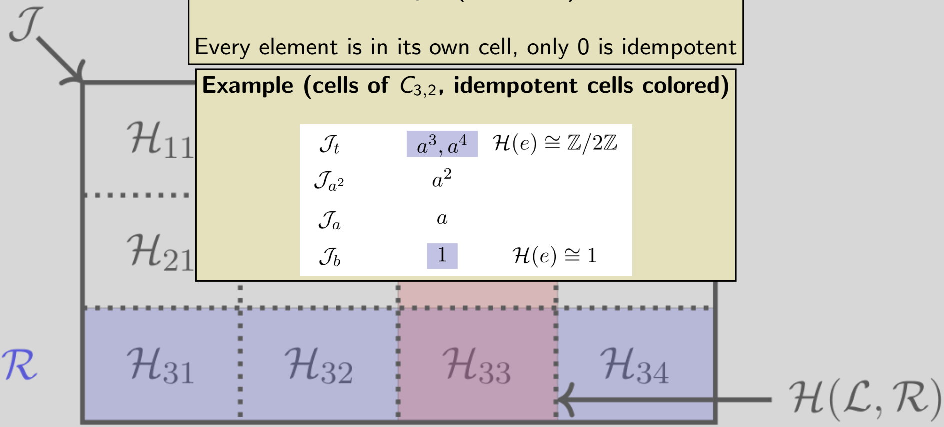

Example (cells of $\mathbb{N}$)

Every element is in its own cell, only 0 is idempotent

Example (cells of $C_{3,2}$, idempotent cells colored)

$\mathcal{J}_t$	a^3, a^4	$\mathcal{H}(e) \cong \mathbb{Z}/2\mathbb{Z}$
$\mathcal{J}_{a^2}$	a^2	
$\mathcal{J}_a$	a	
$\mathcal{J}_b$	1	$\mathcal{H}(e) \cong 1$

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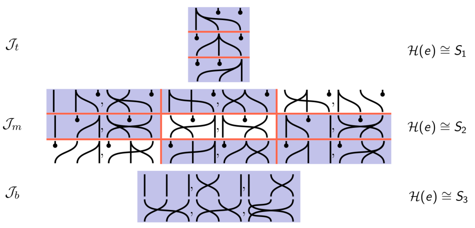
Example (cells of T_3, idempotent cells colored; more in a second)



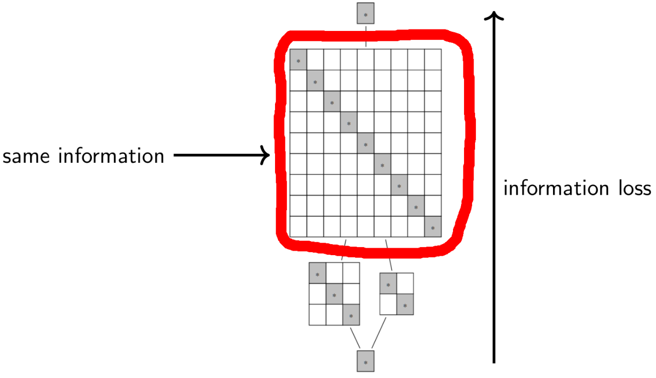
$\mathcal{J}$ $\mathcal{H}_{11}$ $\mathcal{H}_{21}$ $\mathcal{R}$ $(\mathcal{L}, \mathcal{R})$

- Eac… $\mathcal{H}(e)$
- Each $\mathcal{H}(e)$ is a maximal subgroup No internal information loss

The theory of monoids (Green ∼1950++)

$\mathcal{J}$

Computing these "egg box diagrams" is one of the main tasks of monoid theory



$\mathcal{R}$ $\mathcal{R})$

GAP can do these calculations for you (package semigroups)

- Each $\mathcal{H}$ contains no or 1 idempotent e; every e is contained in some $\mathcal{H}(e)$
- Each $\mathcal{H}(e)$ is a maximal subgroup No internal information loss

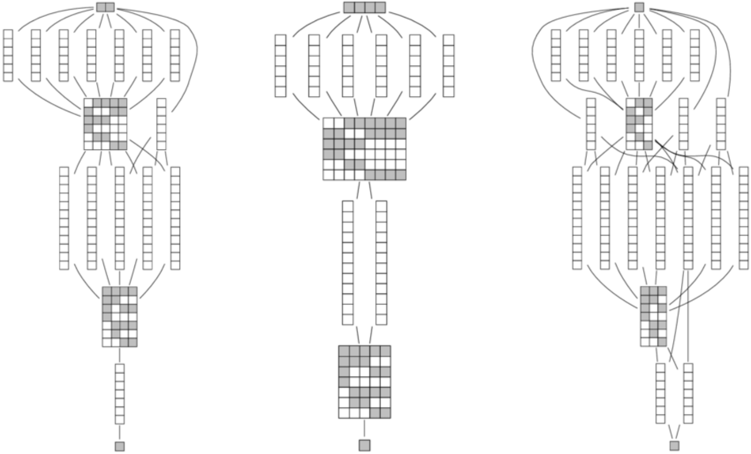
The theory of monoids (Green ∼1950++)

$\mathcal{J}$

$\mathcal{R}$

$\mathcal{L}, \mathcal{R})$

Examples (no specific monoids)

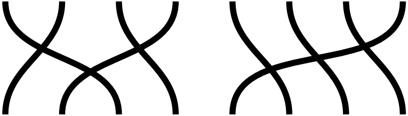


Grey boxes are idempotent H-cells

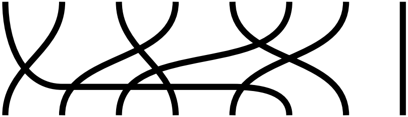
- Ea ... $\mathcal{H}(e)$
- Each $\mathcal{H}(e)$ is a maximal subgroup No internal information loss

Cells of some diagram monoids

Connect eight points at the bottom with eight points at the top:

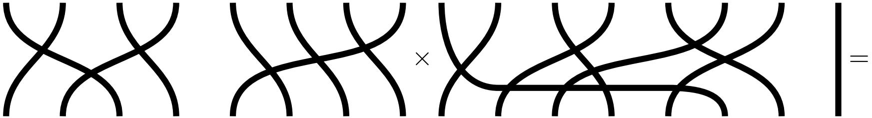
(24138567) $\leftrightsquigarrow$ 

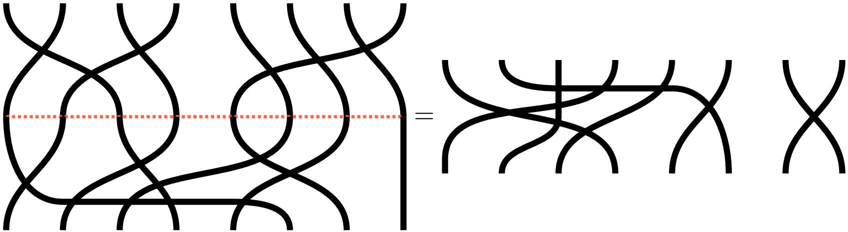
or

(24637158) $\leftrightsquigarrow$ 

We just invented the symmetric group S_8 on $\{1, ..., 8\}$

Cells of some diagram monoids

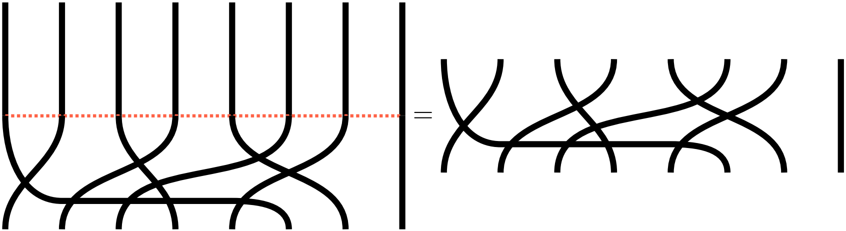




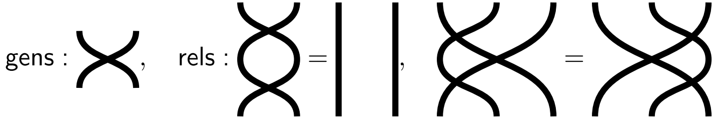
My multiplication rule for gh is "stack g on top of h"

Cells of some diagram monoids

- We clearly have $g(hf) = (gh)f$
- There is a do nothing operation $1g = g = g1$



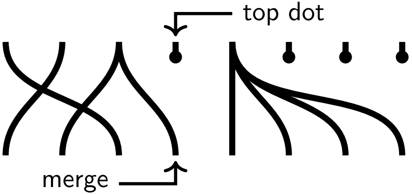
- Generators–relations (the Reidemeister moves), e.g.



Cells of some diagram monoids

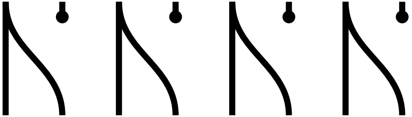
Allow merges and top dots:

(23135555) ⇝



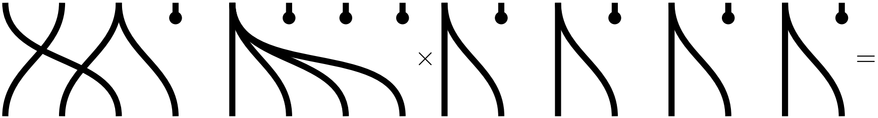
or

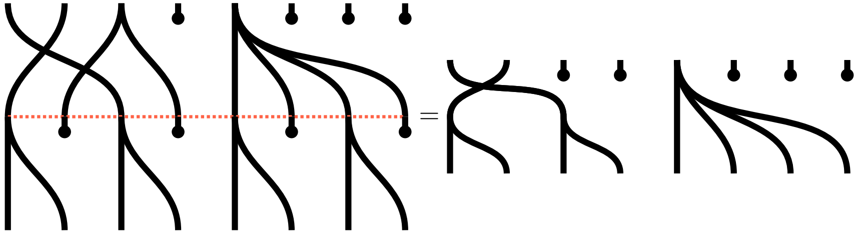
(11335577) ⇝



We just invented the transformation monoid T_8 on $\{1, ..., 8\}$

Cells of some diagram monoids

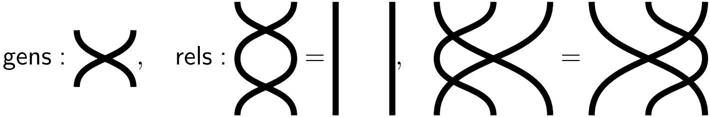




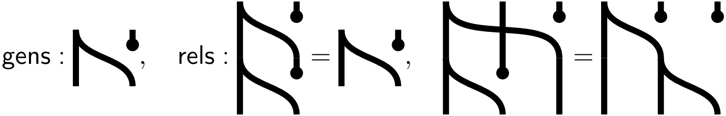
My multiplication rule for gh is "stack g on top of h"

Cells of some diagram monoids

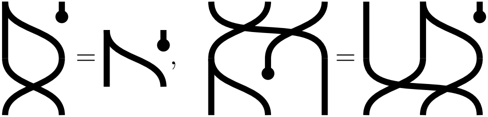
- Generators–relations for $S_n \subset T_n$ (the Reidemeister moves), e.g.



- Generators–relations for the non-invertible part of T_n, e.g.



- Interactions, e.g.



Cells of so

- Genera

Example (cells of T_3, idempotent cells colored)

$\mathcal{J}_t$		(111)		$\mathcal{H}(e) \cong S_1$
		(222)		
		(333)		
$\mathcal{J}_m$	$(122),(211)$	$(121),(212)$	$(221),(112)$	$\mathcal{H}(e) \cong S_2$
	$(133),(311)$	$(313),(131)$	$(113),(331)$	
	$(233),(322)$	$(323),(232)$	$(223),(332)$	
$\mathcal{J}_b$		$(123),(213),(132)$ $(231),(312),(321)$		$\mathcal{H}(e) \cong S_3$

- Generators–relations for the non-invertible part of T_n, e.g.

- Interactions, e.g.

- Genera
- Generators–relations for the non-invertible part of T_n, e.g.

g

- Interact

Example (cells of T_3, idempotent cells colored)

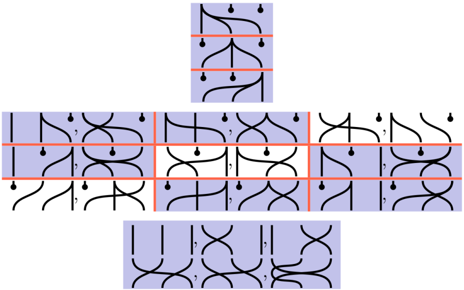
$\mathcal{J}_t$		(111) (222) (333)		$\mathcal{H}(e) \cong S_1$
$\mathcal{J}_m$	(122), (211)	(121), (212)	(221), (112)	$\mathcal{H}(e) \cong S_2$
	(133), (311)	(313), (131)	(113), (331)	
	(233), (322)	(323), (232)	(223), (332)	
$\mathcal{J}_b$		(123), (213), (132) (231), (312), (321)		$\mathcal{H}(e) \cong S_3$

Example (cells of T_3, idempotent cells colored)

$\mathcal{J}_t$ — $\mathcal{H}(e) \cong S_1$

$\mathcal{J}_m$ — $\mathcal{H}(e) \cong S_2$

$\mathcal{J}_b$ — $\mathcal{H}(e) \cong S_3$



Cells of some diagram monoids

Theorem (folklore)

$\mathcal{J}$-cells of T_n are given and ordered by through strands λ

All $\mathcal{J}$-cells contain idempotents

$\mathcal{L}$-cells correspond to fixed bottom ($\left\{ {n \atop \lambda} \right\}$ many), $\mathcal{R}$-cells to fixed top ($\binom{n}{\lambda}$ many)

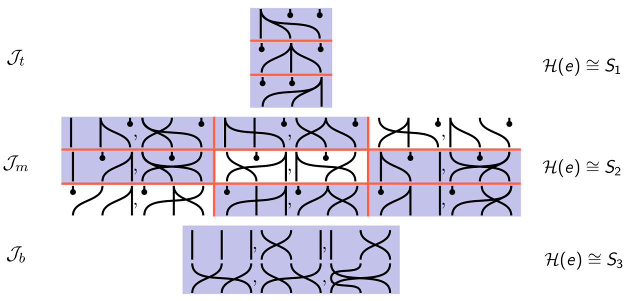
$\mathcal{H}(e) \cong S_\lambda$ for $\lambda = \#$ through strands

- Generators–relations for the non-invertible part of T_n, e.g.

g

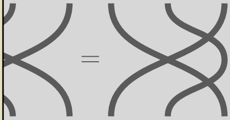
- Interact

Example (cells of T_3, idempotent cells colored)

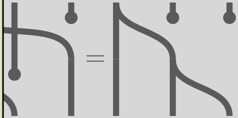


Cells of some diagram m

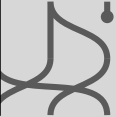
- Generators–relations fo r moves), e.g.

gens : , 

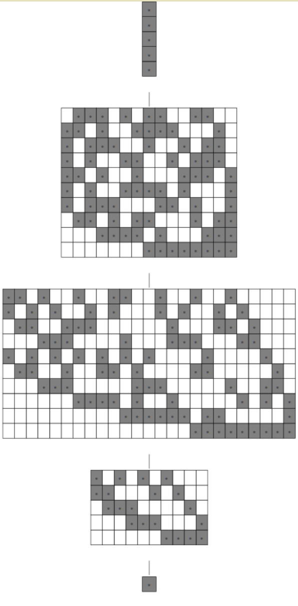
- Generators–relations fo T_n, e.g.

gens : , 

- Interactions, e.g.

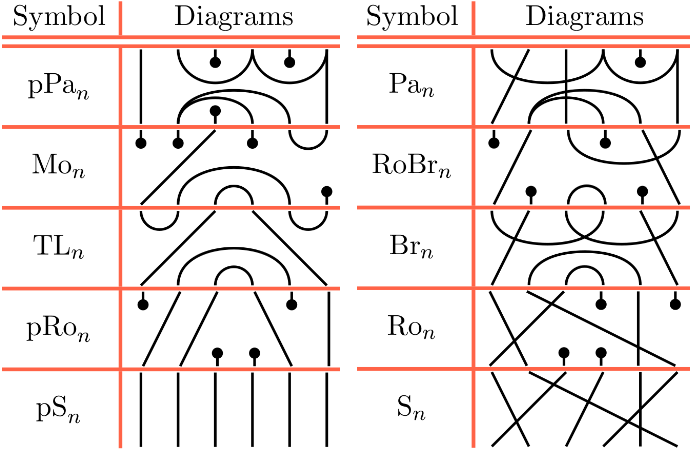
 

Example (T_5 via GAP)



Cells of some diagram monoids

More examples (details on the exercise sheets)
Planar (left) and symmetric (right) diagram monoids, e.g.



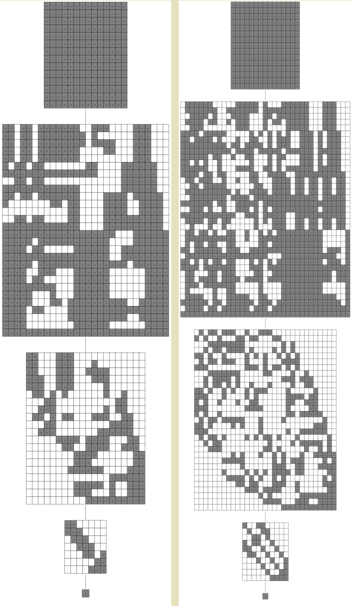
The (planar) symmetric groups $\mathrm{pS}_n, \mathrm{S}_n$ are groups $\Rightarrow$ Boring cells

Cells of som

More examp

Planar (left)

Examples ((planar) partition monoid $\mathrm{pPa}_4, \mathrm{Pa}_4$ via GAP)

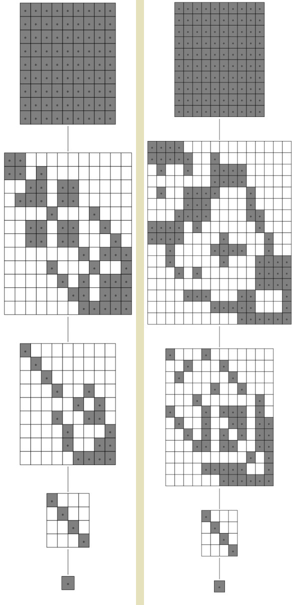


The (planar) symmetric groups $\mathrm{pS}_n, \mathrm{S}_n$ are groups $\Rightarrow$ boring cells

Cells of

More ex
Planar (l

Examples (Motzkin + rook Brauer monoid $\mathrm{Mo}_4, \mathrm{RoBr}_4$ via GAP)



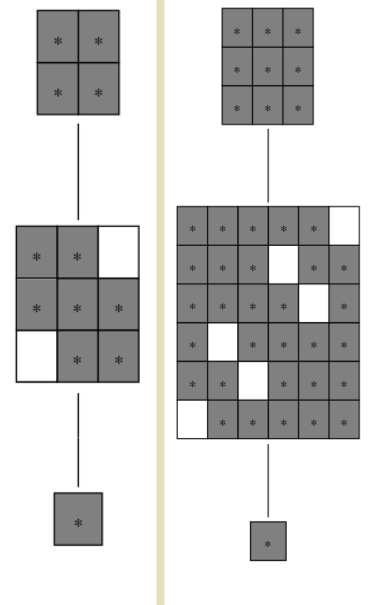
The (pl

Cells of

More ex

Planar (l

Examples (Temperley–Lieb + Brauer monoid $\mathrm{TL}_4, \mathrm{Br}_4$ via GAP)

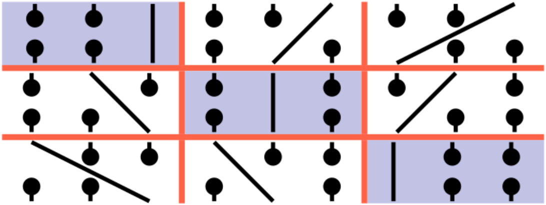


The (pla

More examples (details on the exercise sheets)

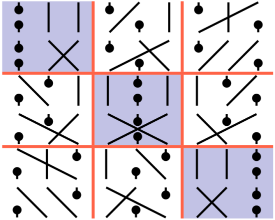
Examples ((planar) rook monoid $\mathrm{pRo}_3, \mathrm{Ro}_3$ by hand)

$\mathcal{J}_1$



$\mathcal{H}(e) \cong 1$

$\mathcal{J}_2$



$\mathcal{H}(e) \cong \mathrm{S}_2$

The (planar) symmetric groups $\mathrm{pS}_n, \mathrm{S}_n$ are groups $\Rightarrow$ Boring cells

The simple reps of monoids

$\phi\colon S \to \mathrm{GL}(V)$ S-representation on a $\mathbb{K}$-vector space V, S is some monoid

- A $\mathbb{K}$-linear subspace $W \subset V$ is S-invariant if $S \centerdot W \subset W$ Substructure
- $V \neq 0$ is called simple if $0, V$ are the only S-invariant subspaces Elements
- Careful with different names in the literature: S-invariant $\leftrightsquigarrow$ subrepresentation, simple $\leftrightsquigarrow$ irreducible
- A crucial goal of representation theory

Find the periodic table of simple S-representations

Chemistry	Group theory	Rep theory
Matter	Groups	Reps
Elements	Simple groups	Simple reps
Simpler substances	Jordan–Hölder theorem	Jordan–Hölder theorem
Periodic table	Classification of simple groups	Classification of simple reps

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Frobenius $\sim$1895++ and others

For groups and $\mathbb{K} = \mathbb{C}$ rep theory is really satisfying

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What about monoids?

Me: Probably not much better than general algebra rep theory...
Jeez, was I wrong...

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Clifford, Munn, Ponizovskiĭ$\sim$1940++ and others

MATRIX REPRESENTATIONS OF COMPLETELY SIMPLE SEMIGROUPS.* 1942

By A. H. Clifford.

ON SEMIGROUP ALGEBRAS

By W. D. Munn

Received 21 July 1954

О матричных представлениях ассоциативных систем *

И. С. Понизовский (Кемерово) 1956

The rep theory of monoids is much better than expected!

Che… M… Ele… Simpler … eorem Perio… ple reps

The simple reps of monoids

Clifford, Munn, Ponizovskiĭ $\sim$1940++ *H*-**reduction**

There is a one-to-one correspondence

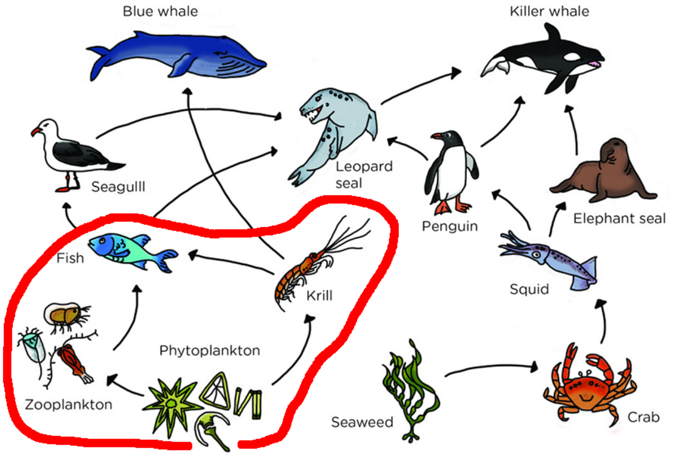
$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}$$

Reps of monoids are controlled by $\mathcal{H}(e)$-cells

- Each simple has a unique maximal $\mathcal{J}(e)$ whose $\mathcal{H}(e)$ does not kill it Apex
- In other words (smod means the category of simples):

$$S\text{-}\mathrm{smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-}\mathrm{smod}$$

Example (anti apex predator)



"Apex = fish" means that the red bubble does not annihilate your rep and the rest does

J-reduction = existence of apexes

Basically, there is a monoid $S_{\mathcal{J}}$ associated to fish with

Simples of $S_{\mathcal{J}} \overset{1:1}{\longleftrightarrow}$ simples of S with apex fish

Clifford, Munn, Ponizovskiĭ $\sim$**1940++** *H*-**reduction**

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Example (groups)

Groups have only one cell – the group itself

H-reduction is trivial for groups

Re ... ls

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The simple reps

Clifford, Munn,

There is a one-to

$$\left\{ \begin{array}{c} \text{simple} \\ \text{apex} \end{array} \quad \text{(any)} \\ \mathcal{J}(e) \right\}$$

Example (cells of $C_{3,2}$, idempotent cells colored)

$\mathcal{J}_t$	a^3, a^4	$\mathcal{H}(e) \cong \mathbb{Z}/2\mathbb{Z}$
$\mathcal{J}_{a^2}$	a^2	
$\mathcal{J}_a$	a	
$\mathcal{J}_b$	1	$\mathcal{H}(e) \cong 1$

Three simple reps over $\mathbb{C}$:
one for $\mathcal{J}_b$ and two for $\mathcal{J}_t$

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Three simple reps over $\mathbb{C}$:
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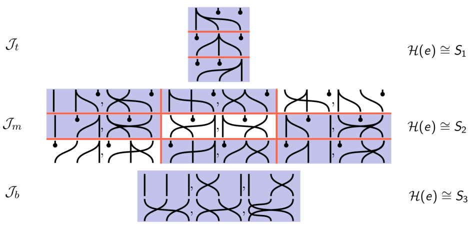
Example (cells of T_3, idempotent cells colored)



Six simple reps over $\mathbb{C}$:
three for $\mathcal{J}_b$, two for $\mathcal{J}_m$ and one for $\mathcal{J}_t$

- Each simple ... ot kill it Apex
- In other word

The simple reps of monoids

Clifford, Munn, Ponizovskiĭ ∼1940++ *H*-**reduction**

There is a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{simple} \\ \text{apex} \end{array} \right.$$

(any) $\mathcal{J}(e)$ $\}$

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Trivial rep of 1 induces to $C_{3,2}$ and has apex $\mathcal{J}_b$

$\mathcal{J}_a, \mathcal{J}_{a^2}, \mathcal{J}_t$ act by zero

Trivial rep of $\mathbb{Z}/2\mathbb{Z}$ induces to $C_{3,2}$ and has apex $\mathcal{J}_t$

Nothing acts by zero

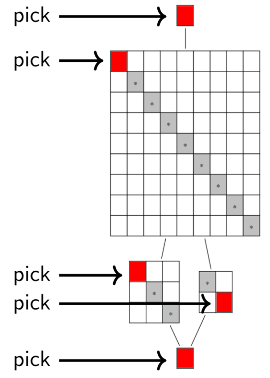
- Each simple ... ot kill it Apex
- In other wor

$S\text{-smod}_{\mathcal{J}(e)} \simeq \mathcal{H}(e)\text{-smod}$

Clif

The

Example (no specific monoid)



Five apexes: bottom cell, big cell, 2x2 cell, 3x3 cell, top cell
Simples for the 2x2 cell are acted on as zero by elements from 3x3 cell, top cell

H-reduction It is sufficient to pick one $\mathcal{H}(e)$ per block

ex

The simple reps of monoids

Clifford, Munn, Ponizovskiĭ ∼1940++ *H*-reduction

- There are cell representations

Cells can be considered S-representations, called *cell representations* or Schützenberger representations, up to higher order terms:

Lemma 3B.1. *Each left cell $\mathcal{L}$ of* S *gives rise to a left* S*-representation* $\Delta_{\mathcal{L}} = \mathbb{K}\mathcal{L}$ *by*

$$a \centerdot l \in \Delta_{\mathcal{L}} = \begin{cases} al & \text{if } al \in \mathcal{L}, \\ 0 & \text{else.} \end{cases}$$

Similarly, right cells give right representations ${}_{\mathcal{R}}\Delta$ *and* J*-cells give birepresentations (often called bimodules). We have* $\dim_{\mathbb{K}}(\Delta_{\mathcal{L}}) = |\mathcal{L}|$ *and* $\dim_{\mathbb{K}}({}_{\mathcal{R}}\Delta) = |\mathcal{R}|$.

- There is a sandwich matrix which takes values in the H-cells
- There is an isomorphism of rings

$$[S\text{-mod}] \cong \prod_{\mathcal{J}(e)} [\mathcal{H}(e)\text{-mod}]$$

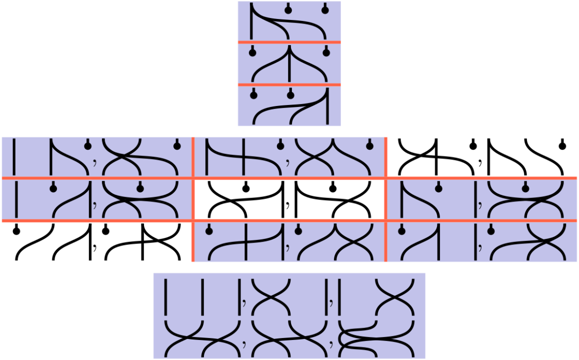
- S is semisimple if and only if all J-cells are idempotent and square, all $\mathcal{H}(e)$ are semisimple + a condition on cell representations
- Many more...

The simple reps of some diagram monoids

$\mathcal{J}_t$ $\qquad$ $\mathcal{H}(e) \cong S_1$

$\mathcal{J}_m$ $\qquad$ $\mathcal{H}(e) \cong S_2$

$\mathcal{J}_b$ $\qquad$ $\mathcal{H}(e) \cong S_3$

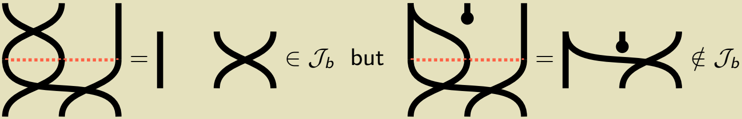


- The transformation monoid T_3 has three apexes, five left cell modules $\Delta(\lambda, i)$, seven right cell modules $\nabla(\lambda, i)$
- Over $\mathbb{C}$ we find 3+2+1 simple modules

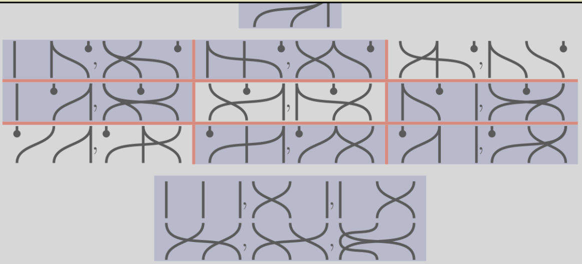
The si

The bottom cell

$\Delta(b)$ is the regular rep of S_3 inflated to T_3:



$\mathcal{J}_m$ $\qquad$ $\mathcal{H}(e) \cong S_2$



$\mathcal{J}_b$ $\qquad$ $\mathcal{H}(e) \cong S_3$

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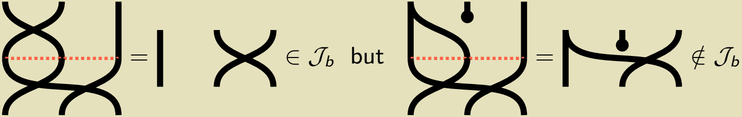
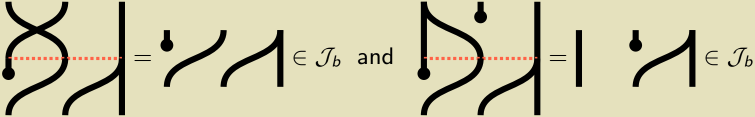
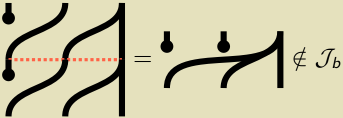


The middle cell, left column (the others are similar)

$\Delta(m, 1)$ is the regular rep of S_2 induced to T_3:





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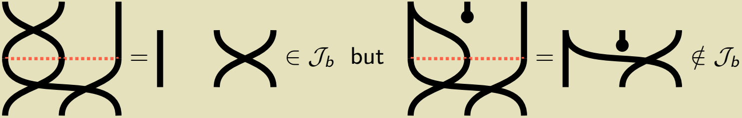

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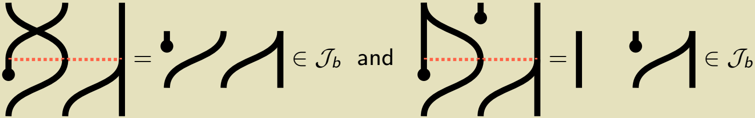
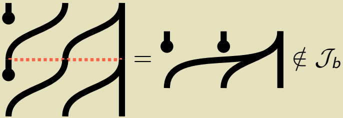




- ▶ The transformation monoid T_3 has three apexes, five left cell modules $\Delta(\lambda, i)$, seven right cell

The top cell

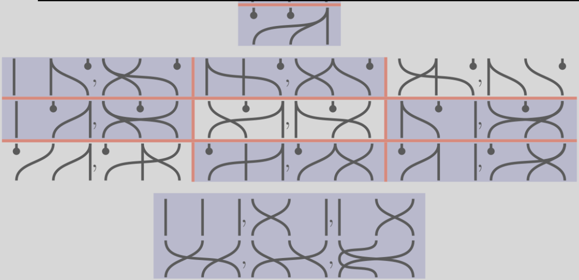
$\Delta(t)$ is the regular rep of S_1 induced to T_3

- ▶ Over $\mathbb{C}$ we find

The simple reps of some diagram monoids

The bottom cell over $\mathbb{C}$

$\Delta(b)$ contributes three simple of S_3 that inflate to T_3
dims are 1, 2, 1 as T_3 reps



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The simple reps of some diagram monoids

$\mathcal{J}_t$

The bottom cell over $\mathbb{C}$

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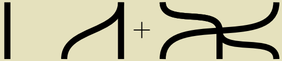
dims are 1, 2, 1 as T_3 reps

$\mathcal{H}(e) \cong S_1$

The middle cell over $\mathbb{C}$

$\Delta(b, 1)$ contributes two simple of S_2 that induce to T_3 (one of them decomposes), e.g.



is an S_2-invariant vector + track its image $\rightsquigarrow$ simple
dims are 3, 2 as T_3 reps

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The simple reps of some diagram monoids

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$\mathcal{J}_t$

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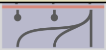

The middle cell over $\mathbb{C}$

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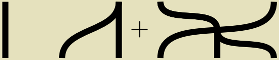


is an S_2-invariant vector + track its image ⇝ simple
dims are 3, 2 as T_3 reps

The top cell

$\Delta(t)$ contributes the trivial T_3 module
dim is 1 as T_3 rep

- The transformatio ... ft cell modules $\Delta(\lambda, i)$, seven right cell mo ...
- Over $\mathbb{C}$ we find 3+2+1 simple modules

The simpl

Sandwich matrices for the middle cell

$$S^{m,triv} = \begin{pmatrix} e_{triv} & e_{triv} & e_{triv} & e_{triv} & 0 & 0 \\ e_{triv} & e_{triv} & e_{triv} & e_{triv} & 0 & 0 \\ e_{triv} & e_{triv} & 0 & 0 & e_{triv} & e_{triv} \\ e_{triv} & e_{triv} & 0 & 0 & e_{triv} & e_{triv} \\ 0 & 0 & e_{triv} & e_{triv} & e_{triv} & e_{triv} \\ 0 & 0 & e_{triv} & e_{triv} & e_{triv} & e_{triv} \end{pmatrix}$$

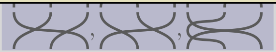
$$S^{m,sign} = \begin{pmatrix} e_{sign} & -e_{sign} & e_{sign} & -e_{sign} & 0 & 0 \\ -e_{sign} & e_{sign} & -e_{sign} & e_{sign} & 0 & 0 \\ e_{sign} & -e_{sign} & 0 & 0 & -e_{sign} & e_{sign} \\ -e_{sign} & e_{sign} & 0 & 0 & e_{sign} & -e_{sign} \\ 0 & 0 & -e_{sign} & e_{sign} & -e_{sign} & e_{sign} \\ 0 & 0 & e_{sign} & -e_{sign} & e_{sign} & -e_{sign} \end{pmatrix}$$

Ranks are 3 and 2 = dims of simples

$\mathcal{J}_t$ $\cong S_1$

$\mathcal{J}_m$ $\cong S_2$

$\mathcal{J}_b$



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$$S^{m,sign} = \begin{pmatrix} e_{sign} & -e_{sign} & e_{sign} & -e_{sign} & 0 & 0 \\ -e_{sign} & e_{sign} & -e_{sign} & e_{sign} & 0 & 0 \\ e_{sign} & -e_{sign} & 0 & 0 & -e_{sign} & e_{sign} \\ -e_{sign} & e_{sign} & 0 & 0 & e_{sign} & -e_{sign} \\ 0 & 0 & -e_{sign} & e_{sign} & -e_{sign} & e_{sign} \\ 0 & 0 & e_{sign} & -e_{sign} & e_{sign} & -e_{sign} \end{pmatrix}$$

Ranks are 3 and 2 = dims of simples

$\mathcal{J}_t$ $\cong S_1$

$\mathcal{J}_m$ $\cong S_2$

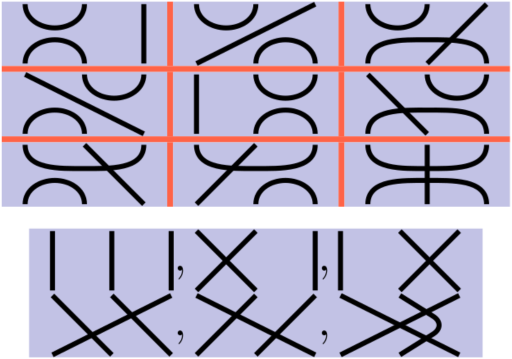
$\mathcal{J}_b$ $(e) \cong S_3$

Theorem (folklore)

The simple T_n-reps are $L(\lambda, K)$ for K a simple S_λ-rep
Unless K is the sign S_λ-rep the induction to the cell is simple
For $K =$ sign the $L(\lambda, K)$ are of dimension $\binom{n-1}{\lambda-1}$

- The tra ules $\Delta(\lambda, i)$, seven right cell modules $\nabla(\lambda, i)$
- Over $\mathbb{C}$ we find 3+2+1 simple modules

The simple reps of some diagram monoids



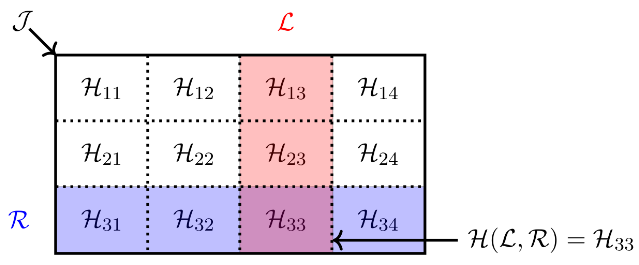
$\mathcal{J}_1$: $\mathcal{H}(e) \cong \mathrm{S}_1$

$\mathcal{J}_3$: $\mathcal{H}(e) \cong \mathrm{S}_3$

- The Brauer monoid Br_3 has two apexes, four left/right cell modules
- Over $\mathbb{C}$ we find $3+1$ simple modules
- Other diagram algebras are similar; more on the exercise sheets

Summary

H-reduction reduces monoid rep theory to group rep theory

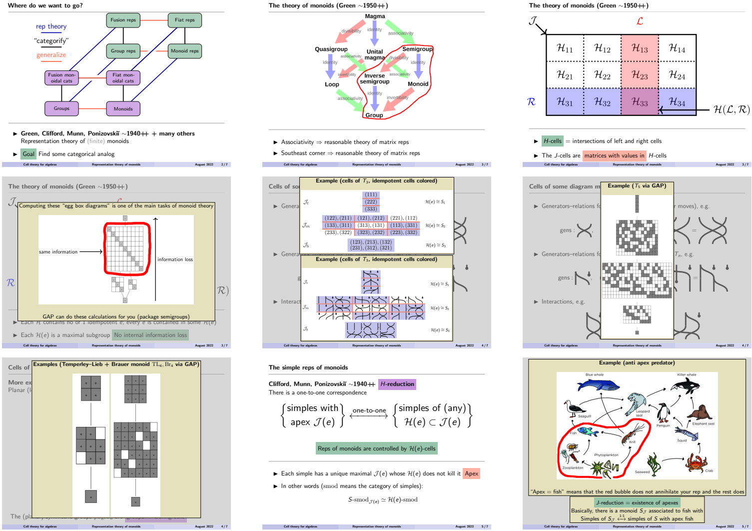


Clifford, Munn, Ponizovskiĭ ∼1940++ ($\mathcal{H}$-reduction)

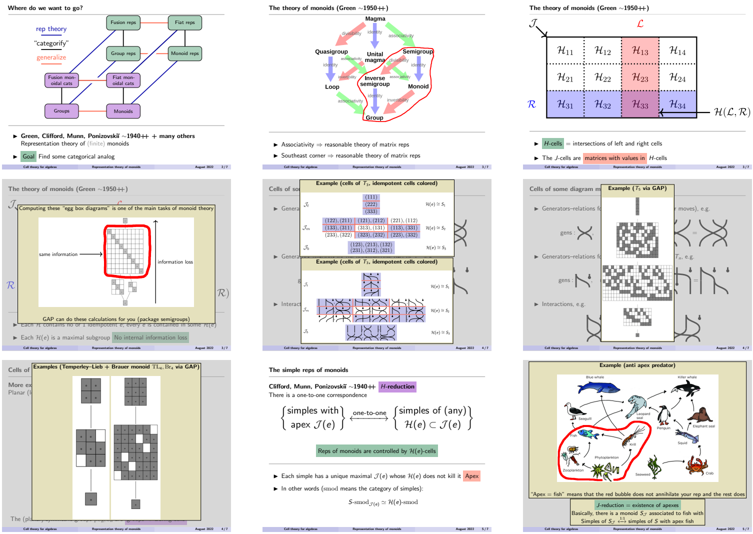
There is a one-to-one correspondence

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}$$

Reps of monoids are controlled by $\mathcal{H}(e)$ cells



There is still much to do...



Thanks for your attention!