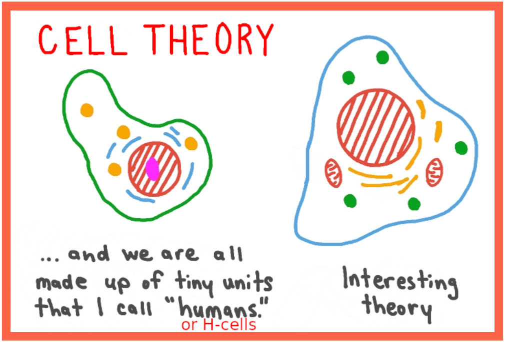
Representation theory of algebras

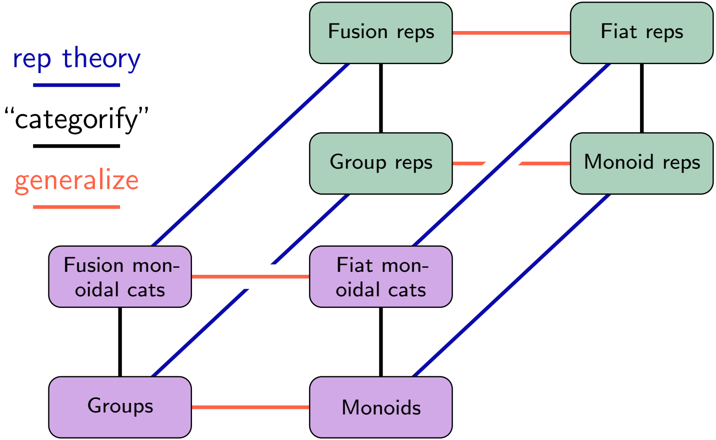
Or: Cell theory for algebras

Daniel Tubbenhauer



Part 1: Reps of monoids; Part 3: Reps of monoidal cats

Where do we want to go?



- **Green, Clifford, Munn, Ponizovskiĭ ∼1940++ + many others**
  Representation theory of (finite) monoids
- Goal Find some categorical analog

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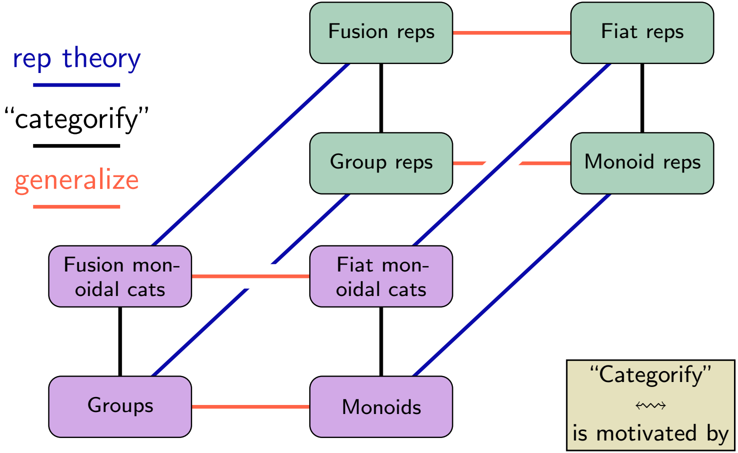


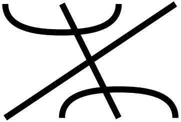
- **Green, Clifford, Munn, Ponizovskiĭ ∼1940++ + many others**
  Representation theory of (finite) monoids
- Goal Find some categorical analog

Where do we want to go?

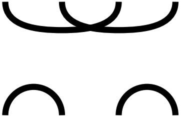
- Today: off track Interlude on H-reduction for algebras
- We will discover the H-reduction for algebras in real time!
- Examples we discuss The good, the ugly and the bad

The good

Connect 4 points at the bottom with 4 points at the top, potentially turning back:

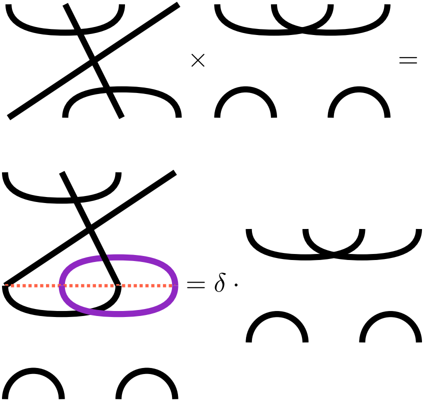
$\{\{1,-4\},\{2,4\},\{3,-2\},\{-1,-3\}\} \leftrightsquigarrow$ 

or

$\{\{1,2\},\{3,4\},\{-1,-3\},\{-2,-4\}\} \leftrightsquigarrow$ 

We just invented the Brauer monoid Br_4 on $\{1,...,4\} \cup \{-1,...,-4\}$

The good



Fix some field $\mathbb{K}$ and $\delta \in \mathbb{K}$, evaluate circles to $\delta \Rightarrow$ Brauer algebra $Br_4(\delta)$
The Brauer monoid is the non-linear version of $Br_4(1)$

The good

Clifford, Munn, Ponizovskiĭ ∼1940++ ***H*-reduction**

There is a one-to-one correspondence

$$\begin{Bmatrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{Bmatrix} \xleftrightarrow{\text{one-to-one}} \begin{Bmatrix}\text{simples of (any)}\\ \mathcal{H}(e)\subset\mathcal{J}(e)\end{Bmatrix}$$

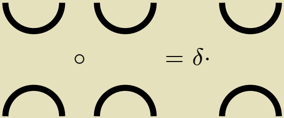
Reps of monoids are controlled by $\mathcal{H}(e)$-cells

- $Br_n(\delta)$ is not a monoid $\Rightarrow$ H-reduction does not apply
- My favorite strategy Ignore problems and just go for it

The good

Crucial notion in the linear world:

Pseudo idempotent $ee = \delta e$ for $\delta \neq 0$



Clifford, Munn, Poni

There is a one-to-one

$$\left\{\begin{matrix}\text{simples w} \\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \quad \left\{\begin{matrix}\text{s of (any)} \\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}$$

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Clifford, Munn, Poni

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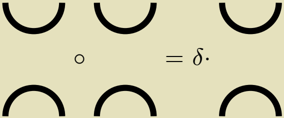
$\left\{\begin{array}{c}\text{simples w} \\ \text{apex } \mathcal{J}(e)\end{array}\right\}$ ... s of (any) ... $\mathcal{H}(e) \subset \mathcal{J}(e)$ $\}$

Crucial notion in the linear world:

Pseudo idempotent $ee = \delta e$ for $\delta \neq 0$



Reps of monoids are controlled by $\mathcal{H}(e)$-cells

- $Br_n($
- My

Pseudo idempotent can be **normalized** to idempotents

$$ee = \delta e \Rightarrow \tfrac{e}{\delta}\tfrac{e}{\delta} = \tfrac{e}{\delta}$$

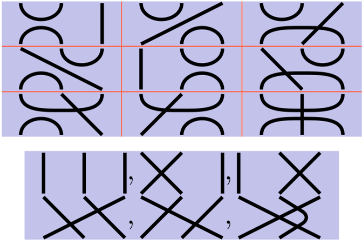
Green, Clifford, Munn, Ponizovskiĭ $\Rightarrow$ pseudo idempotents are good!

The go

Example (cells of $Br_3(\delta)$ for $\delta \neq 0$, pseudo idempotent cells colored)

$\mathcal{J}_1$ — $\mathcal{H}(e) \cong S_1$

$\mathcal{J}_3$ — $\mathcal{H}(e) \cong S_3$



Clifford

There i

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The go

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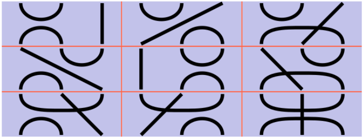
- M

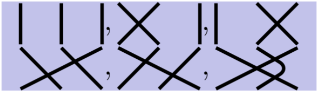
Example (cells of $Br_3(\delta)$ for $\delta \neq 0$, pseudo idempotent cells colored)

$\mathcal{J}_1$



$\mathcal{H}(e) \cong S_1$

$\mathcal{J}_3$



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Example (cells of $Br_3(\delta)$ for $\delta = 0$, pseudo idempotent cells colored)

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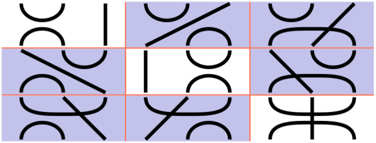
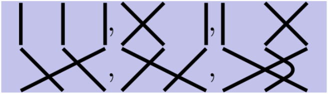


$\mathcal{H}(e) \cong S_1$

$\mathcal{J}_3$



$\mathcal{H}(e) \cong S_3$

The g

Cliffo

There

Theorem (Brown ∼1953, Fishel–Grojnowski ∼1995)

H-reduction works for the Brauer algebra

CANONICAL BASES FOR THE BRAUER CENTRALIZER ALGEBRA

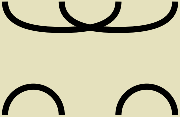
S. FISHEL AND I. GROJNOWSKI

(a) J-cells $\leftrightsquigarrow$ through strands, all J-cells are pseudo idempotent for $\delta \neq 0$ and $\mathcal{H}(e)$ are symmetric groups in the through strands

(b) For $\delta = 0$ similar but with $\mathcal{J}_0$ having no pseudo idempotent

- $Br_n(\delta)$ is not a
- My favorite st

The top cell:



etc. are nilpotent for $\delta = 0$

The good

Clifford,

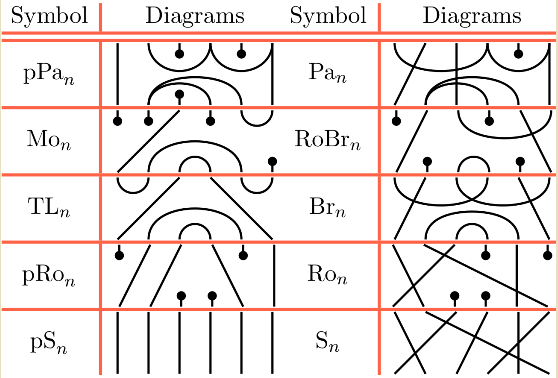
There is

$\{ S \ldots) \}$

- $Br_n($
- My

Theorem (Brown ∼1953, Fishel–Grojnowski ∼1995 and folklore)

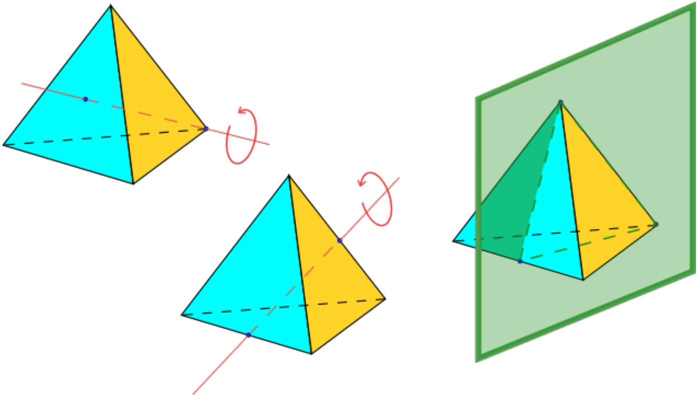
H-reduction works for other diagram algebra and quantum versions



More details on the exercise sheets

The ugly

S_4 :



- Symmetric groups S_n of order $n!$ are the symmetry groups of $(n-1)$ simplices
- Frobenius $\sim$1895++ Their (complex) rep theory is well understood
- They are groups $\Rightarrow$ Green's cell theory is boring

The ugly

S_4 :

```
-----------------------
Class |   1  2  3  4  5
Size  |   1  3  6  8  6
Order |   1  2  2  3  4
-----------------------
p  =  2   1  1  1  4  2
p  =  3   1  2  3  1  5
-----------------------
X.1    +  1  1  1  1  1
X.2    +  1  1 -1  1 -1
X.3    +  2  2  0 -1  0
X.4    +  3 -1 -1  0  1
X.5    +  3 -1  1  0 -1
```

, S_5 :

```
-----------------------------
Class |   1  2  3  4  5  6  7
Size  |   1 10 15 20 30 24 20
Order |   1  2  2  3  4  5  6
-----------------------------
p  =  2   1  1  1  4  3  6  4
p  =  3   1  2  3  1  5  6  2
p  =  5   1  2  3  4  5  1  7
-----------------------------
X.1    +  1  1  1  1  1  1  1
X.2    +  1 -1  1  1 -1  1 -1
X.3    +  4 -2  0  1  0 -1  1
X.4    +  4  2  0  1  0 -1 -1
X.5    +  5  1  1 -1 -1  0  1
X.6    +  5 -1  1 -1  1  0 -1
X.7    +  6  0 -2  0  0  1  0
```

- **Kazhdan–Lusztig (KL) ∼1979, Lusztig ∼1984, folklore** Use a J-reduction approach
- Lusztig's approach gives an H-reduction way of classifying simple S_n reps
- Aside The character table of S_n is integral ⇒ categorification!

The ugly

The (linear) cell orders and equivalences for fixed basis B:

$$x \leq_L y \Leftrightarrow \exists z : y \Subset zx$$
$$x \leq_R y \Leftrightarrow \exists z' : y \Subset xz'$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z' : y \Subset zxz'$$
$$x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x)$$
$$x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x)$$
$$x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x)$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- *H*-cells = intersections of left and right cells
- Slogan Cells measure information loss

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Left, right and tw ce classes

Green cells in linear

$B = \{x, y, z, \ldots\}$ fix basis of an algebra A

$\Subset$ = has a nonzero coefficient when expressed via B

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Example (groups and monoids)

For B=monoid basis: linear cells = Green cells

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For B=monoid basis: linear cells = Green cells

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Example (group-like)

All invertible basis elements form the smallest cell (well, kind of...)

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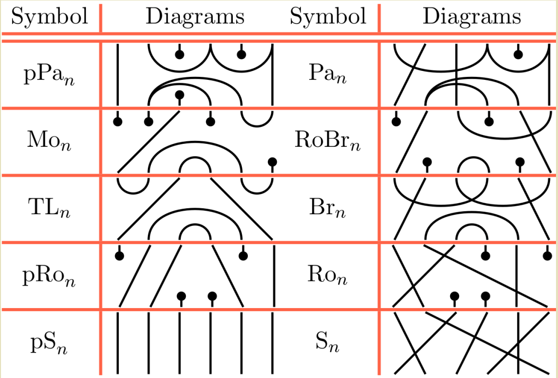
Example (diagram algebras)



For B=diagram basis: linear cells = Green cells

Left, right and t... e classes

- H-cells =
- Slogan Ce

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Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get algebras $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ by killing higher order terms
- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

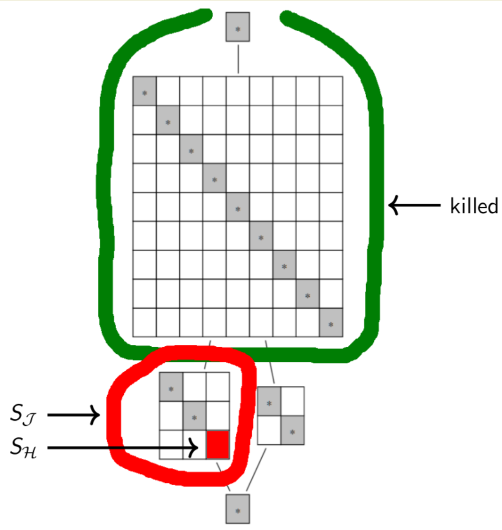
The ugly

The (linear) cell o

Left, right and tw ... ence classes

- Get algebras
- Pseudo idemp

Example (no specific algebra)



The ugly

Example (cells for $S_2 = \langle 1 \rangle$ and $S_3 = \langle 1, 2 \rangle$ over $\mathbb{C}$, pseudo idempotent cells colored)

$\mathcal{J}_{w_0}$	b_1	$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{C}$

$\mathcal{J}_{w_0}$	b_{121}		$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_{m}$	b_1	b_{12}	$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{C}$
	b_{21}	b_2	
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$		$\mathrm{S}_{\mathcal{H}} \cong \mathbb{C}$

For B=KL basis b_w, $\cong_s$ = up to scaling (more on the exercise sheets)

- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

Example (cells for $S_2 = \langle 1 \rangle$ and $S_3 = \langle 1,2 \rangle$ over $\mathbb{Z}/2\mathbb{Z}$, pseudo idempotent cells colored)

$\mathcal{J}_{w_0}$	b_1	
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$

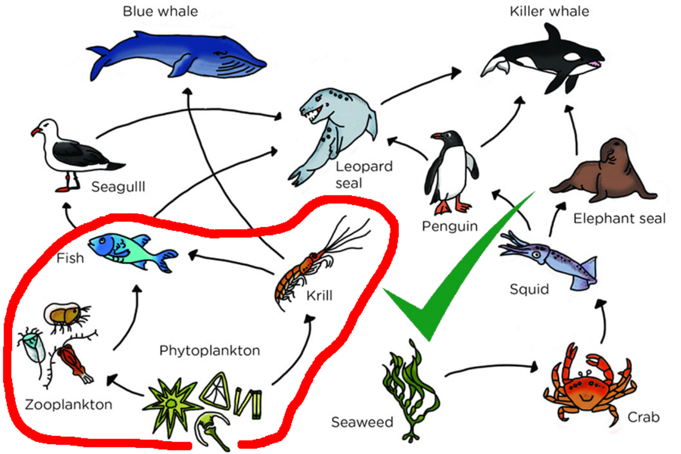
$\mathcal{J}_{w_0}$	b_{121}	
$\mathcal{J}_{m}$	$\begin{matrix} b_1 & b_{12} \\ b_{21} & b_2 \end{matrix}$	$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{K}$
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{K}$

For B=KL basis b_w, $\cong_s$ = up to scaling (more on the exercise sheets)

- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

Theorem (KL ∼1979, Lusztig ∼1984, folklore)

The notion of an apex makes sense for the pair (S_n, B_{KL}) *J*-reduction



"Apex = fish" means that the KL basis elements in the red bubble do not annihilate your rep and the others do

Theorem (KL ∼1979, Lusztig ∼1984, folklore)

For S_n there exists a basis B such that:

(a) All simple S_n modules have a unique apex

(b) For $\mathrm{char}(\mathbb{K}) = 0$, J-cells $\leftrightsquigarrow$ partitions of n, all J-cells are pseudo idempotent and all corresponding $S_{\mathcal{H}}$ are isomorphic to $\mathbb{K}$

(c) H-reduction works for S_n

$$x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x)$$

$$x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x)$$

$$x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x)$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get algebras $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ by killing higher order terms
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$$x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x)$$

For the record

(b) works over any field but with p-restricted partitions
There is also a quantum version

Left, right and ... e classes

- Get algebras $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ by killing higher order terms
- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

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$$x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x)$$

Left, right ... sses

▶ Get al

▶ Pseud

Example (symmetric group S_3)

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

$\mathcal{J}_{w_0}$	b_{121}	$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_m$	b_1, b_{12} / b_{21}, b_2	$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_\emptyset$	$b_\emptyset$	$\mathrm{S}_{\mathcal{H}} \cong \mathbb{C}$

The simple reps over $\mathbb{C}$ corresponding to the three $\mathcal{J}(e)$ cells via

$$\chi_2 \leftrightsquigarrow \mathcal{J}_\emptyset,\ \chi_3 \leftrightsquigarrow \mathcal{J}_m,\ \chi_1 \leftrightsquigarrow \mathcal{J}_{w_0},$$

Theorem (KL ∼1979, Lusztig ∼1984, folklore)

For S_n there exists a basis B such that:

(a) All simple S_n modules have a unique apex

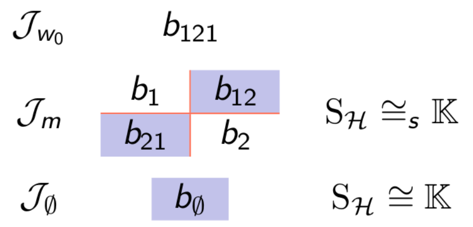
(b) For $\mathrm{char}(\mathbb{K}) = 0$, J-cells $\leftrightsquigarrow$ partitions of n, all J-cells are pseudo idempotent and all corresponding $S_{\mathcal{H}}$ are isomorphic to $\mathbb{K}$

(c) H-reduction works for S_n

$$x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x)$$

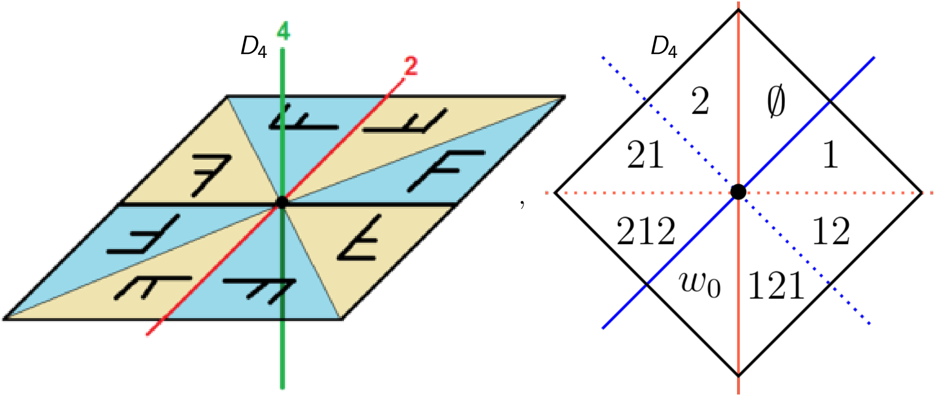
Example (symmetric group S_3)

```
------------------
Class |    1     3
Size  |    1     2
Order |    1     3
------------------
p  =  2    1     3
p  =  3    1     1
------------------
X.1   +    1     1
X.2   +    1 -   1
X.3   +    2    -1
```



The simple reps over $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 2$ corresponding to the two $\mathcal{J}(e)$ cells via $\chi_1 = \chi_2 \leftrightsquigarrow \mathcal{J}_\emptyset$, $\chi_3 \leftrightsquigarrow \mathcal{J}_m$,

The bad



- Dihedral groups D_n of order $2n$ are the symmetry groups of n gons
- Frobenius ∼1895++ Their (complex) rep theory is well understood
- They are groups ⇒ Green's cell theory is boring

The bad

D_4 :

```
------------------------
Class |   1  2  3  4  5
Size  |   1  1  2  2  2
Order |   1  2  2  2  4
------------------------
p  =  2   1  1  1  1  2
------------------------
X.1    +   1  1  1  1  1
X.2    +   1  1 -1  1 -1
X.3    +   1  1  1 -1 -1
X.4    +   1  1 -1 -1  1
X.5    +   2 -2  0  0  0
```

, D_5 :

```
-------------------------
Class |   1  2     3     4
Size  |   1  5     2     2
Order |   1  2     5     5
-------------------------
p  =  2   1  1     4     3
p  =  5   1  2     1     1
-------------------------
X.1    +   1  1     1     1
X.2    +   1 -1     1     1
X.3    +   2  0    Z1 Z1#2
X.4    +   2  0  Z1#2   Z1
```

- **Kazhdan–Lusztig (KL) ∼1979, Lusztig ∼1984, folklore** Use a J-reduction approach
- Lusztig's approach almost gives an H-reduction way of classifying simple D_n reps
- Aside The character table of D_n is not integral ⇒ looks bad for categorification

The bad

The (linear) cell orders and equivalences for a fixed basis B:

$$x \leq_L y \Leftrightarrow \exists z : y \Subset zx$$
$$x \leq_R y \Leftrightarrow \exists z' : y \Subset xz'$$
$$x \leq_{LR} y \Leftrightarrow \exists z, z' : y \Subset zxz'$$
$$x \sim_L y \Leftrightarrow (x \leq_L y) \wedge (y \leq_L x)$$
$$x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x)$$
$$x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x)$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- H-cells = intersections of left and right cells
- Slogan Cells measure information loss

The bad

The (linear) cell orders and equivalences for fixed basis B:

$$
\begin{aligned}
x \leq_L y &\Leftrightarrow \exists z : y \Subset zx \\
x \leq_R y &\Leftrightarrow \exists z' : y \Subset xz' \\
x \leq_{LR} y &\Leftrightarrow \exists z, z' : y \Subset zxz' \\
x \sim_L y &\Leftrightarrow (x \leq_L y) \wedge (y \leq_L x) \\
x \sim_R y &\Leftrightarrow (x \leq_R y) \wedge (y \leq_R x) \\
x \sim_{LR} y &\Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x)
\end{aligned}
$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get algebras $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ by killing higher order terms
- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

The bad

The

Example (cells for $D_4, D_5 = \langle 1,2 \rangle$ over $\mathbb{C}$, pseudo idempotent cells colored)

D_4:

$\mathcal{J}_{w_0}$	b_{1212}		$S_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_m$	b_1, b_{121}	b_{12}	$S_{\mathcal{H}} \cong_s \mathbb{C}$
	b_{21}	b_2, b_{212}	$S_{\mathcal{H}} \cong_s \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$		$S_{\mathcal{H}} \cong \mathbb{C}$

D_5:

$\mathcal{J}_{w_0}$	b_{12121}		$S_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_m$	b_1, b_{121}	b_{12}, b_{1212}	$S_{\mathcal{H}} \cong_s \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$
	b_{21}, b_{2121}	b_2, b_{212}	
$\mathcal{J}_{\emptyset}$	$b_{\emptyset}$		$S_{\mathcal{H}} \cong \mathbb{C}$

For B=KL basis b_w, $\cong_s$ = up to scaling (more on the exercise sheets)

Left

- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

The bad

The (linear) cell orders and equivalences for fixed basis B:

$$x \leq_L y \Leftrightarrow \exists z : y \Subset zx$$

Theorem (Lusztig ∼80ish, folklore)

For D_n, n odd there exists a basis B such that:

(a) All simple D_n modules have a unique apex

(b) There are three J-cells, all J-cells are pseudo idempotent and all corresponding $S_{\mathcal{H}}$ are isomorphic to $\mathbb{C}$ or $\mathbb{C}[\mathbb{Z}/(\frac{n-1}{2})\mathbb{Z}]$

There is also a slightly more involved version for general $\mathbb{K}$

(c) H-reduction works for D_n

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get algebras $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ by killing higher order terms
- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

Example (dihedral group D_5)

```
------------------------
Class |   1  2    3    4
Size  |   1  5    2    2
Order |   1  2    5    5
------------------------
p  =  2   1  1    4    3
p  =  5   1  2    1    1
------------------------
X.1   +   1  1    1    1
X.2   +   1 -1    1    1
X.3   +   2  0   Z1 Z1#2
X.4   +   2  0 Z1#2   Z1
```

$\mathcal{J}_{w_0}$	b_{12121}		$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_m$	b_1, b_{121}	b_{12}, b_{1212}	$\mathrm{S}_{\mathcal{H}} \cong_s \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$
	b_{21}, b_{2121}	b_2, b_{212}	
$\mathcal{J}_\emptyset$	$b_\emptyset$		$\mathrm{S}_{\mathcal{H}} \cong \mathbb{C}$

The simple reps over $\mathbb{C}$ corresponding to the three $\mathcal{J}(e)$ cells via

$\chi_2 \leftrightsquigarrow \mathcal{J}_\emptyset,\ \chi_3, \chi_4 \leftrightsquigarrow \mathcal{J}_m,\ \chi_1 \leftrightsquigarrow \mathcal{J}_{w_0},$

$$x \sim_R y \Leftrightarrow (x \leq_R y) \wedge (y \leq_R x)$$

$$x \sim_{LR} y \Leftrightarrow (x \leq_{LR} y) \wedge (y \leq_{LR} x)$$

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get algebras $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ by killing higher order terms
- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

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p  =  2   1  1    4    3
p  =  5   1  2    1    1
-------------------------
X.1   +   1  1    1    1
X.2   +   1 -1    1    1
X.3   +   2  0   Z1 Z1#2
X.4   +   2  0 Z1#2   Z1
```

$\mathcal{J}_{w_0}$	b_{12121}		$S_{\mathcal{H}} \cong_s \mathbb{C}$
$\mathcal{J}_m$	b_1, b_{121}	b_{12}, b_{1212}	$S_{\mathcal{H}} \cong_s \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$
	b_{21}, b_{2121}	b_2, b_{212}	
$\mathcal{J}_\emptyset$	$b_\emptyset$		$S_{\mathcal{H}} \cong \mathbb{C}$

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Example (dihedral group D_5)

```
-------------------------
Class |   1       3    4
Size  |   1       2    2
Order |   1       5    5
-------------------------
p  =  2   1       4    3
p  =  5   1       1    1
-------------------------
X.1   +   1       1    1
X.2   +   1 -     1    1
X.3   +   2      Z1 Z1#2
X.4   +   2    Z1#2   Z1
```

Z1=golden ratio

$\mathcal{J}_{w_0}$	b_{12121}		
$\mathcal{J}_m$	b_1, b_{121}	b_{12}, b_{1212}	$S_{\mathcal{H}} \cong \frac{\mathbb{K}[X]}{X^2+X+1}$
	b_{21}, b_{2121}	b_2, b_{212}	
$\mathcal{J}_\emptyset$	$b_\emptyset$		$S_{\mathcal{H}} \cong \mathbb{K}$

The simple reps over $\mathbb{K}$ with $\mathrm{char}(K) = 2$ corresponding to the two $\mathcal{J}(e)$ cells via

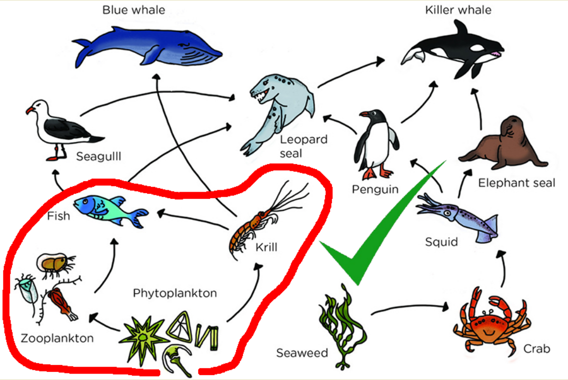
$$\chi_2 \leftrightsquigarrow \mathcal{J}_\emptyset,\ \chi_3, \chi_4 \leftrightsquigarrow \mathcal{J}_m,\ \chi_1 \leftrightsquigarrow \mathcal{J}_{w_0},$$

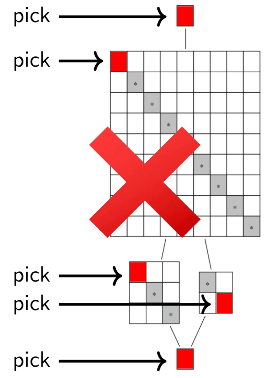
Th

Th

Theorem (Lusztig ∼80ish, folklore)

The J-reduction (notion of an apex) makes sense for all Coxeter groups





But H-reduction only works in type A and odd dihedral type

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

H-reduction	Powerful, but doesn't work in general
J-reduction	General, but not so powerful

- Get algebras
- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

The bad

J-reduction often doesn't get you far

A middle cell in type E_8 where $\dim S_{\mathcal{J}} = 202671840$ and zillions of associated simples:

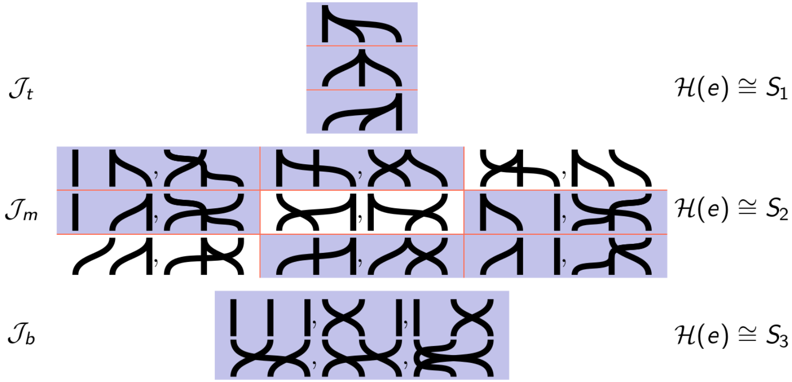
$\mathcal{J}_{23}$	$\mathbf{7}_{420,420}$	$\mathbf{5}_{756,420}$	$\mathbf{6}_{1596,420}$	$\mathbf{5}_{168,420}$	$\mathbf{3}_{378,420}$	$\mathbf{4}_{1092,420}$	$\mathbf{2}_{70,420}$
	$\mathbf{5}_{420,756}$	$\mathbf{8}_{756,756}$	$\mathbf{7}_{1596,756}$	$\mathbf{7}_{168,756}$	$\mathbf{8}_{378,756}$	$\mathbf{8}_{1092,756}$	$\mathbf{7}_{70,756}$
	$\mathbf{6}_{420,1596}$	$\mathbf{7}_{756,1596}$	$\mathbf{12}_{1596,1596}$	$\mathbf{8}_{168,1596}$	$\mathbf{9}_{378,1596}$	$\mathbf{13}_{1092,1596}$	$\mathbf{11}_{70,1596}$
	$\mathbf{5}_{420,168}$	$\mathbf{7}_{756,168}$	$\mathbf{8}_{1596,168}$	$\mathbf{12}_{168,168}$	$\mathbf{7}_{378,168}$	$\mathbf{12}_{1092,168}$	$\mathbf{12}_{70,168}$
	$\mathbf{3}_{420,378}$	$\mathbf{8}_{756,378}$	$\mathbf{9}_{1596,378}$	$\mathbf{7}_{168,378}$	$\mathbf{15}_{378,378}$	$\mathbf{14}_{1092,378}$	$\mathbf{19}_{70,378}$
	$\mathbf{4}_{420,1092}$	$\mathbf{8}_{756,1092}$	$\mathbf{13}_{1596,1092}$	$\mathbf{12}_{168,1092}$	$\mathbf{14}_{378,1092}$	$\mathbf{21}_{1092,1092}$	$\mathbf{24}_{70,1092}$
	$\mathbf{2}_{420,70}$	$\mathbf{7}_{756,70}$	$\mathbf{11}_{1596,70}$	$\mathbf{12}_{168,70}$	$\mathbf{19}_{378,70}$	$\mathbf{24}_{1092,70}$	$\mathbf{39}_{70,70}$

$A_{k,l}$ = H-cells of size A arranged in a $(k \times l)$-matrix

Left, right and two-sided cells (a.k.a. L, R and J-cells): equivalence classes

- Get algebras $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ by killing higher order terms
- Pseudo idempotents make $S_{\mathcal{J}}$, $S_{\mathcal{H}}$ unital

The good, the ugly and the bad – comparison



- For monoids all H-cells within one J-cell are of the same size
- For the good, the ugly and the odd bad the same is true
- For the even bad this is false

The

Strategy (Brown $\sim$1953, König–Xi $\sim$1999, folklore)

GENERALIZED MATRIX ALGEBRAS

W. P. BROWN

THE SEMISIMPLICITY OF ω_f^n*

By William P. Brown

(Received December 6, 1953)
(Revised November 15, 1954)

CELLULAR ALGEBRAS: INFLATIONS AND MORITA EQUIVALENCES

STEFFEN KÖNIG and CHANGCHANG XI

Find axioms ensuring that
H-cells within one J-cell are of the same size

- For the good, the ugly and the odd bad the same is true
- For the even bad this is false

The good, the ugly and the bad – comparison

Sandwich datum:

- *A partial ordered set* $\Lambda = (\Lambda, \leq_\Lambda)$
- *finite sets* M_λ *(bottom) and* N_λ *(top) for all* $\lambda \in \Lambda$
- *an algebra* $\mathbb{S}_\lambda$ *and a fixed basis* B_λ *of it for all* $\lambda \in \Lambda$
- *a* $\mathbb{K}$*-basis* $\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D \in M_\lambda, U \in N_\lambda, b \in \mathrm{B}_\lambda\}$ *of* A

$\mathbb{S}_\lambda$ = sandwiched algebras

A sandwich cellular algebra A: sandwich datum + some axioms such as

$$xc^\lambda_{D,b,U} \equiv \sum_{S \in M_\lambda, a \in B} r(S,D) \cdot c^\lambda_{S,a,U} + \text{higher order friends}$$

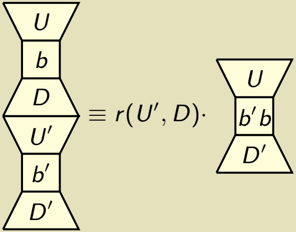
Cellular = all $\mathbb{S}_\lambda \cong \mathbb{K}$ = all H-cells are of size one

The good, the ugly a…

Sandwich datum:

- *A partial order…*
- *finite sets M_λ (…ll $\lambda \in \Lambda$*
- *an algebra $\mathbb{S}_\lambda$ …r all $\lambda \in \Lambda$*
- *a $\mathbb{K}$-basis $\{c^\lambda_{D,b,U} \mid \lambda \in \Lambda, D \in M_\lambda, U \in N_\lambda, b \in \mathrm{B}_\lambda\}$ of A*

Approximate picture to keep in mind



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The good, the ugly a…

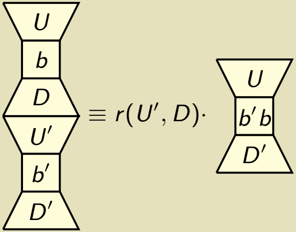
Sandwich datum:

- *A partial order*
- *finite sets* M_λ (… *all* $\lambda \in \Lambda$
- *an algebra* $\mathbb{S}_\lambda$ … *r all* $\lambda \in \Lambda$
- *a* $\mathbb{K}$ … $\}$ *of* A

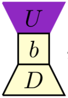
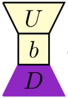
$\mathbb{S}_\lambda =$ sa…

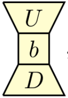
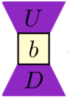
A sandw…

Approximate picture to keep in mind



H-cells are of equal size, by definition! As free $\mathbb{K}$ vector spaces:

$\mathcal{L}(\lambda, U) \cong M_\lambda \otimes_{\mathbb{K}} \mathbb{S}_\lambda \leftrightsquigarrow$  , $\mathcal{R}(\lambda, D) \cong \mathbb{S}_\lambda \otimes_{\mathbb{K}} N_\lambda \leftrightsquigarrow$  ,

$\mathcal{J}_\lambda \cong M_\lambda \otimes_{\mathbb{K}} \mathbb{S}_\lambda \otimes_{\mathbb{K}} N_\lambda \leftrightsquigarrow$  , $\mathcal{H}_{\lambda,D,U} \cong \mathbb{S}_\lambda \leftrightsquigarrow$  .

Cellular = all $\mathbb{S}_\lambda \cong \mathbb{K}$ = all H-cells are of size one

The good, the ugly and the bad – comparison

In spirit of Clifford, Munn, Ponizovskiĭ ∼1940++ ***H*-reduction**

There is a one-to-one correspondence (under some conditions on $\mathbb{K}$ and $\mathbb{S}_\lambda$)

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \lambda\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of}\\ \mathbb{S}_\lambda\end{matrix}\right\}$$

Reps are controlled by the sandwiched algebras

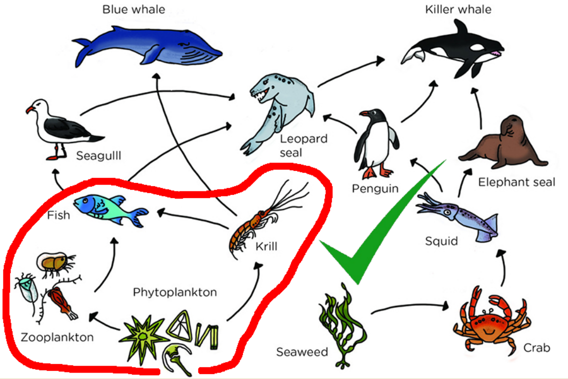
- Each simple has a unique maximal $\lambda \in \Lambda$ where having a pseudo idempotent is replaced by a paring condition Apex
- In other words (smod means the category of simples):

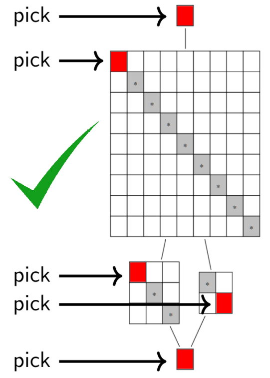
$$S\text{-smod}_\lambda \simeq \mathbb{S}_\lambda\text{-smod}$$

The good, the ugly and the bad – comparison

For sandwich cellular algebras

The J-reduction (notion of an apex) makes sense
The H-reduction works





$$S\text{-smod}_\lambda \simeq \mathbb{S}_\lambda\text{-smod}$$

The good, the ugly and the bad – comparison

In spi

There

Example

All algebras are sandwich cellular with $\Lambda = \{\bullet\}$ and $\mathbb{S}_\bullet = A$

We get the fantastic *H*-reduction tautology:

$$\left\{\begin{matrix}\text{simples of}\\ A\end{matrix}\right\} = \left\{\begin{matrix}\text{simples with}\\ \text{apex } \bullet\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of}\\ \mathbb{S}_\bullet\end{matrix}\right\} = \left\{\begin{matrix}\text{simples of}\\ A\end{matrix}\right\}$$

The point is to find a good sandwich datum!

- E... tent is re...
- In...

The good, the ugl

Example (cf. talk 1)

Many monoid algebras with the monoid basis

In spirit of Clifford, Munn, Ponizovskii ∼1940++ ***H*-reduction**

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The good, the ugl

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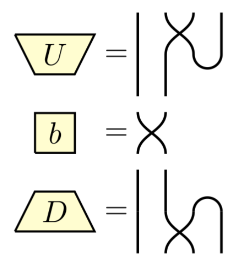
In spirit of Clifford, Munn, Ponizovskiĭ ∼1940++ $\mathcal{H}$-reduction

There is a one-to-one ... s on $\mathbb{K}$ and $\mathbb{S}_\lambda$)

Example

Diagram algebras with the diagram basis

The good



$$\left\{ \begin{matrix} \text{simple} \ldots \text{ples of} \\ \text{ape} \ldots \mathbb{S}_\lambda \end{matrix} \right\}$$

Rep ... bras

- Each simple has a ... pseudo idempotent is replaced by a paring condition Apex
- In other words (smod means the category of simples):

$$S\text{-smod}_\lambda \simeq \mathbb{S}_\lambda\text{-smod}$$

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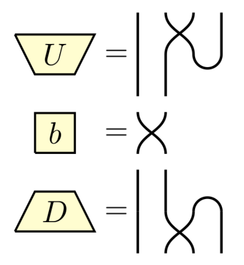
Example

Diagram algebras with the diagram basis

The good



{ simple ... ples of
ape ... $\mathbb{S}_\lambda$ }

Reps ... bras

- Each simple has a ... pseudo idempotent is replaced by a paring condition Apex

Example

S_n with the KL basis

The ugly

(smod means the category of simples):

$$S\text{-smod}_\lambda \simeq \mathbb{S}_\lambda\text{-smod}$$

Example (cf. talk 1)

Many monoid algebras with the monoid basis

In spirit of Clifford, Munn, Ponizovskii ∼1940++ $\mathcal{H}$**-reduction**

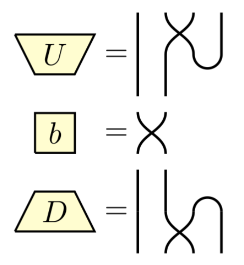
There is a one-to-one ... s on $\mathbb{K}$ and $\mathbb{S}_\lambda$)

$$\left\{\begin{matrix}\text{simple} \ldots \text{ples of} \\ \text{ape} \ldots \mathbb{S}_\lambda\end{matrix}\right\}$$

Example

Diagram algebras with the diagram basis

The good



Reps ... bras

- Each simple has a ... pseudo idempotent is replaced by a paring condition Apex

(sm ... imples):

...nod

Example

S_n with the KL basis

The ugly

Example

D_n, n odd with the KL basis

The odd bad

Example (cf. talk 1)

Many monoid algebras with the monoid basis

In spirit of Clifford, Munn, Ponizovskii ~1940++ $\mathcal{H}$**-reduction**

There is a one-to-one ... s on $\mathbb{K}$ and $\mathbb{S}_\lambda$)

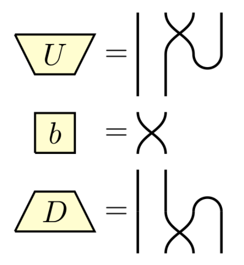
$\left\{ \text{simple} \ldots \text{ples of} \atop \text{ape} \ldots \mathbb{S}_\lambda \right\}$

Reps ... bras

Example

Diagram algebras with the diagram basis

The good



- Each simple has a ... pseudo idempotent is replaced by a paring condition Apex

(sm ... imp ... nod

Example

S_n with the KL basis

The ugly

Example

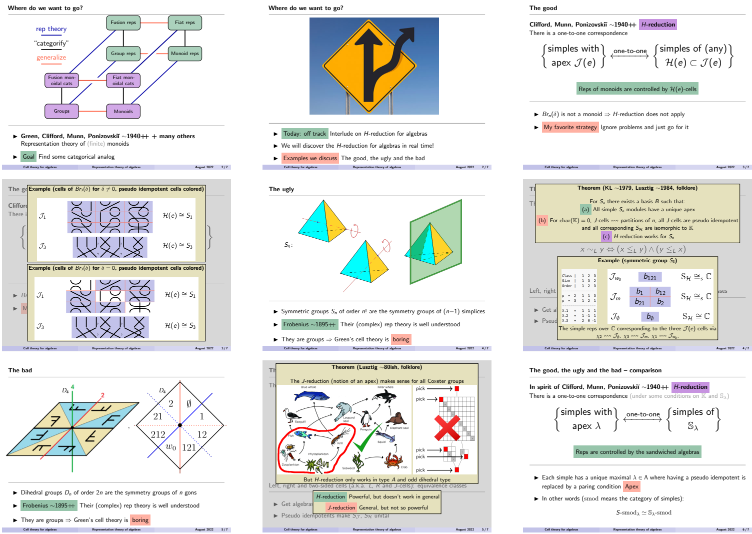
D_n, n odd with the KL basis

The odd bad

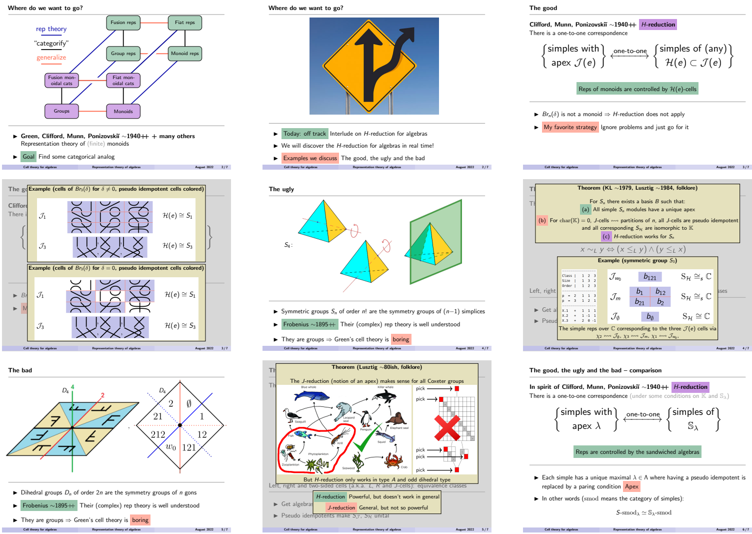
Nonexample

D_n, n even with the KL basis

The even bad



There is still much to do...



Thanks for your attention!