Exercises 1

1. Two trivial representations!?

Let S be a finite monoid.

a) Show that S has a unique bottom and top J-cell.
b) Show that both of these J-cells are idempotent.
c) Let $\mathbb{K}$ be some field and $G \subset S$ be the subgroup of all invertible elements. Then we define trivial representations (yes, a monoid has two trivial representations):

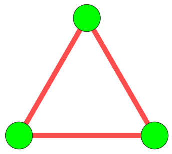
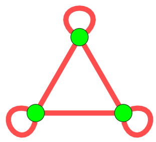
$$M_b\colon S \to \mathbb{K}, s \longmapsto \begin{cases} 1 & \text{if } s \in G, \\ 0 & \text{else,} \end{cases} \qquad M_t\colon S \to \mathbb{K}, s \longmapsto 1.$$

Identify the apexes of the simple S-modules M_b and M_t.

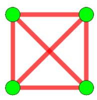
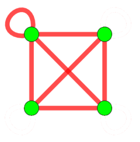
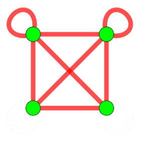
2. Endomorphisms of graphs

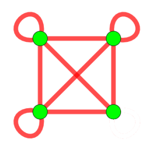
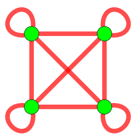
Let Γ be a graph. The set $S = \mathrm{End}(\Gamma)$ of graph endomorphisms is a monoid via composition.

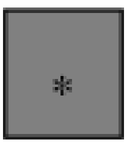
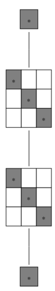
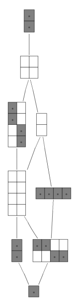
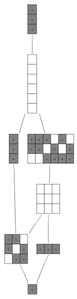
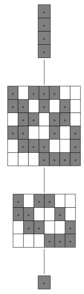
a) Compute the cell structure and classify simple modules for $S = \mathrm{End}(\Gamma)$ and $T = \mathrm{End}(\Gamma')$ for the following two graphs.

$\Gamma =$  , $\Gamma' =$ 

b) If you have done (a), then you should have seen two familiar monoids. Can you guess the general picture how they arise as graph monoids?
c) (*) Here are a few more graph monoids $S_i = \mathrm{End}(\Gamma_i)$ and their cell pictures. I do not know the general pattern; maybe someone has an idea.

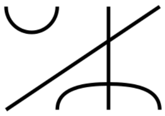
$\Gamma_0 =$  , $\Gamma_1 =$  , $\Gamma_2 =$  ,

$\Gamma_3 =$  , $\Gamma_4 =$ 

For $S_0\colon |S_0| = 24$,  , For $S_1\colon |S_1| = 34$,  , For $S_2\colon |S_2| = 56$,  , For S_3 : $|S_3| = 108$,  , For S_4 : $|S_4| = 256$, 

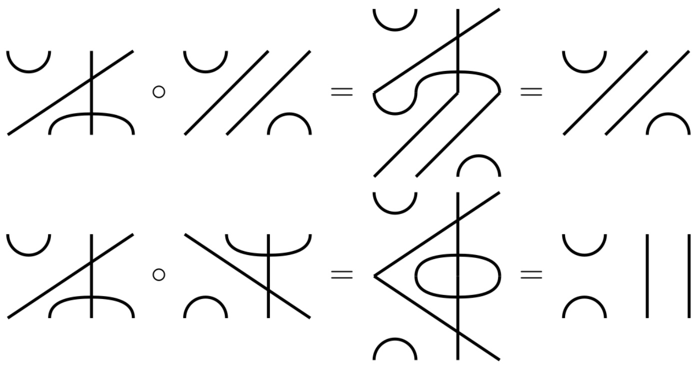
3. Diagram monoids and H-reduction

The Brauer monoid Br_n on n strands is the monoid consisting of all perfect matchings of $\{1, \dots, n\}$ with $\{-1, \dots, -n\}$, which we identify with points in the strip $\mathbb{R} \times [0, 1]$, with n points for $\{1, \dots, n\}$ at the bottom and n points for $\{-1, \dots, -n\}$ at the top line of the strip. For example

 $\leftrightsquigarrow \{\{1, -4\}, \{2, 4\}, \{3, -3\}, \{-1, -2\}\}$

Two diagrams represent the same element if and only if they represent the same perfect matching.

Stacking and removing of internal components defines a multiplication $\circ$ on Br_n, *e.g.*



(Associativity of $\circ$ is not immediate, but also not hard to see.)

a) Compute the L, R and J-cells of Br_3.
b) Compute the idempotent J-cells.
c) Compute the $\mathcal{H}(e)$-cells.
d) Parameterize the simple Br_3-modules.
e) Guess the picture for general n from the one for Br_3.

4. Binomial coefficients

Let $\mathbb{K}$ be an arbitrary field. Let $S = pRo_n$ be the planar rook monoid (the monoid of all planar partitions of $\{1, \dots, n\} \cup \{-1, \dots, -n\}$ with at most two parts and no connections within $\{1, \dots, n\}$ or $\{-1, \dots, -n\}$; see also the remarks).

a) Show that S is semisimple.
b) Compute the dimensions of the simple S-modules.

- There might be typos on the exercise sheets, my bad. Be prepared.
- Star exercises are a bit trickier; prime exercises use notions I haven't explained.