Exercises - hints and remarks 1

GAP installation guide https://www.gap-system.org/Download/index.html
Semigroups https://www.gap-system.org/Packages/semigroups.html
The code

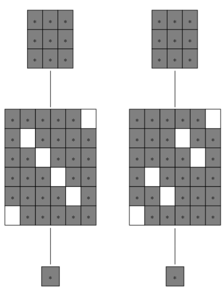
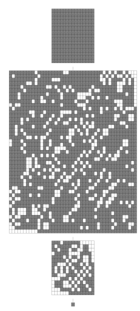
```
LoadPackage(semigroups);
S := FullTransformationMonoid(3);
FileString(t3.dot, DotString(S));
```

produces a dot string diagram of the cells of the transformation monoid on $\{1,2,3\}$. The transformation monoid can be replaced by any other monoid. (Beware: The diagrams GAP produces are flipped top-to-bottom when compared to the conventions of the lecture.)
GAP's presentation of the various monoids is often not optimal. One can vary the presentation, which gives slightly different outputs. For example,

```
LoadPackage(semigroups);
S := BrauerMonoid(4);
FileString(br4.dot, DotString(S));
```

```
LoadPackage(semigroups);
S:=Semigroup(Bipartition([[1,-1],[2,-2],[3,-3],[4,-4]]),
Bipartition([[1,-2],[2,-1],[3,-3],[4,-4]]),
Bipartition([[1,-1],[2,-3],[3,-2],[4,-4]]),
Bipartition([[1,-1],[2,-2],[3,-4],[4,-3]]),
Bipartition([[1,2],[-1,-2],[3,-3],[4,-4]]),
Bipartition([[1,-1],[2,3],[-2,-3],[4,-4]]),
Bipartition([[1,-1],[2,-2],[3,4],[-3,-4]]));
FileString(br4.dot, DotString(S));
```

both produce cells for the Brauer monoid, but the pictures are different:

(1)  , cells of Br_6 produced similarly: 

The arguably the most important diagrams monoids are:

(1) The partition monoid Pa_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$, the planar partition monoid pPa_n is the planar version of Pa_n.

(2) The rook Brauer monoid $RoBr_n$ is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with at most two parts, the Motzkin monoid Mo_n is the planar version of $RoBr_n$.

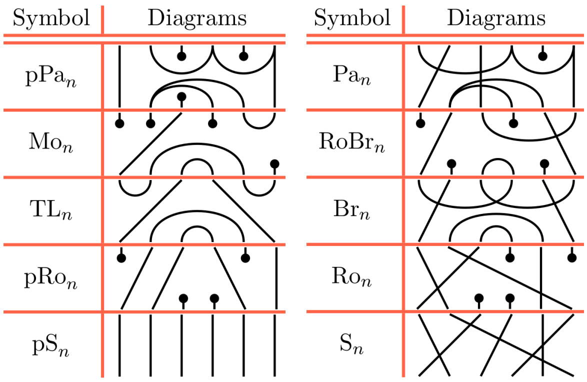
(3) The Brauer monoid Br_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with two parts, the Temperley–Lieb monoid TL_n is the planar version of Br_n.

(4) The rook monoid Ro_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with with at most two parts and no connection within $\{1,\dots,n\}$ or $\{-1,\dots,-n\}$, the planar rook monoid pRo_n is the planar version of Ro_n.

(5) The symmetric group S_n is the monoid of all partitions of $\{1,\dots,n\} \cup \{-1,\dots,-n\}$ with with two parts and no connection within $\{1,\dots,n\}$ or $\{-1,\dots,-n\}$, the planar symmetric group pS_n is the planar version of S_n.

Planar = can be drawn without intersection while not leaving the defining box of endpoints.

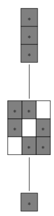
The pictures to keep in mind are the following, with the planar monoids displayed on the left:



Redo the Brauer exercise for your favorite(s) among these diagram monoids; it is fun. (The monoids pS_n and S_n are a bit boring, and my recommendation is to compute the cells for pPa_2 and TL_4.)

Hints for Exercise 2

GAP produces

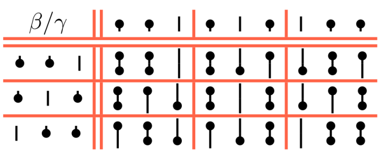
For $S: \ |S| = 3!$,  , For $T: \ |T| = 3^3$, 

Hints for Exercise 3

GAP says that Br_6 has the following statistics. We have four linearly ordered J-cells, 1+15+45+15 L and R cells. The H-cells are of sizes 6,4,2,1, so we have 10395 elements in total. The top cell consists of idempotent H-cells only, and otherwise there is at least one idempotent per J-cell, see Equation 1.

Hints for Exercise 4

For each J-cell there are bottom diagrams $\beta_1,\dots,\beta_L$ and top diagrams $\gamma_1,\dots,\gamma_R$ indexing the rows and columns of the J-cell in question. The Gram matrix is $P_{ij} = 1$ if $\beta_j \circ \gamma_i = 1$ and $P_{ij} = 0$ else. For example,



$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the Gram matrix of the second J-cell of pRo_3. Show that this implies that pRo_n is semisimple. The cell modules are then the simple modules, so their dimensions are easy to compute.