Hints for Exercise 2

The one-dimensional representations are easy to construct. For the two-dimensional representations use

\[
1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 2 \mapsto \begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) \\ -\sin(2\pi k/n) & \cos(2\pi k/n) \end{pmatrix}.
\]

Via easy calculations (seriously: these are 2x2 matrices!) one verifies: The matrices satisfy the relations of \( D_n \) and have no common eigenvector, so the associated representations are simple. They are also nonconjugate for \( k \in \{1, \ldots, \lfloor n-1/2 \rfloor \} \). Finally, the sum of the squares of their dimensions is \( 2n \), so we are done.

In general, \( D_n \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \), and one can use \( 1, 2 \) as the generators of the two groups in this semidirect product. Now induce from those two groups and hope for the best.

Hints for Exercise 3

Unless the characteristic of \( \mathbb{K} \) is two, the picture should look like

\[
\begin{array}{ccc|cc}
J_m & b_{1}, b_{121}, \ldots & b_{12}, b_{1212}, \ldots & S_H \cong \mathbb{K}[\mathbb{Z}] \\
J_0 & b_{21}, b_{2121}, \ldots & b_{2}, b_{212}, \ldots & S_H \cong \mathbb{K}
\end{array}
\]

The Grothendieck algebra (abelian or additive, that does not make a difference) of \( \text{SO}_3(\mathbb{C}) \) can be computed via the SageMath online calculator, see above, with the code

```python
A=WeylCharacterRing(A1,style=coroots);
k=5;
j=4;
A(2*k,0)*A(2*j,0)
```

You need to vary \( k \) and \( j \), and identify \( b_{121} \) with \( A(2) = A(2,0) \) up to scaling. Neither \( b_{121} \) nor \( A(2) \) satisfy any polynomial relation, but both generate the respective algebras.

Hints for Exercise 4

Unless the characteristic of \( \mathbb{K} \) is nonzero and small, the picture for \( n \) being odd should look like

\[
\begin{array}{ccc|cc}
J_{w_0} & b_{w_0} & S_H \cong \mathbb{K} \\
J_m & b_1, b_{121}, \ldots & b_{12}, b_{1212}, \ldots & S_H \cong \mathbb{K}[\mathbb{Z}/n^{-1}\mathbb{Z}] \\
J_0 & b_{21}, b_{2121}, \ldots & b_{2}, b_{212}, \ldots & S_H \cong \mathbb{K}
\end{array}
\]

That the diagonal \( S_H \) have pseudo idempotents is clear by \( b_1b_1 = 2b_1 \). For the off-diagonal elements let us take \( n = 7 \) and \( b = b_{12} - b_{1212} + b_{121212} \). Then the multiplication table
verifies that $b^2 = b$. The general case is similar. (Note that $b$ would be an infinite alternating sum for $n = \infty$, and that is why the off-diagonal $S_H$ do not have pseudo idempotents in the infinite case.)

The isomorphism $S_H \cong \mathbb{K}[\mathbb{Z}/n\mathbb{Z}]$ for nonsilly $\mathbb{K}$ can be verified as follows. Let $U^3_k(X)$ be the (Chebyshev-like multiplication by quantum three) polynomial defined via $U^3_0(X) = 1$, $U^3_1(X) = X$ and

$$U^3_k(X) = (X - 1)U^3_{k-1}(X) - U^3_{k-2}(X).$$

This polynomial is the defining polynomial for $SO_3(\mathbb{C})$ in the sense that $U^3_k(X)$ corresponds to the highest weight summand in the tensor product $(X = \mathbb{C}^3)^\otimes k$. Here is some SageMath code:

```
A=WeylCharacterRing(A1,style=coroots);
    gen=A(2,0);
k=7;
def U(n,x):
    if n == 0:
        return 1
    elif n == 1:
        return x
    else:
        return (x-1)*U(n-1,x) - U(n-2,x)
print(U(k,gen))
```

Now $U^3_m(b_{12}) = 0$ for $m = n-1$, so $S_H \cong \mathbb{K}[X]/(U^3(X))$. Since $U^3_m(X)$ has distinct roots, we can then rescale $\mathbb{K}[X]/(U^3_m(X))$ to $\mathbb{K}[X]/(X^m - 1) \cong \mathbb{K}[\mathbb{Z}/m\mathbb{Z}]$.

That was the case of $SO_3(\mathbb{C})$, so you need to argue why this implies the same for the KL basis of the finite dihedral group.