Exercises - hints and remarks 2

SageMath online calculator `https://sagecell.sagemath.org/` with the relevant material summarized on
`https://doc.sagemath.org/html/en/thematic_tutorials/lie/weyl_character_ring.html`
Magma online calculator `http://magma.maths.usyd.edu.au/calc/`

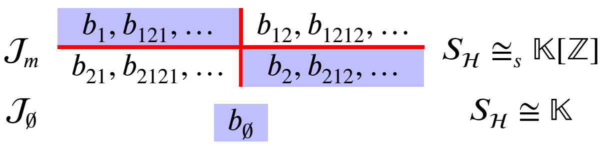
Hints for Exercise 2
The one dimensional representations are easy to construct. For the two dimensional representations use

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 2 \longmapsto \begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) \\ -\sin(2\pi k/n) & -\cos(2\pi k/n) \end{pmatrix}.$$

Via easy calculations (seriously: these are 2x2 matrices!) one verifies: The matrices satisfy the relations of D_n and have no common eigenvector, so the associated representations are simple. They are also nonconjugate for $k \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. Finally, the sum of the squares of their dimensions is $2n$, so we are done.

In general, $D_n \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, and one can use 12 and 1 as the generators of the two groups in this semidirect product. Now induce from those two groups and hope for the best.

Hints for Exercise 3
Unless the characteristic of $\mathbb{K}$ is two, the picture should look like

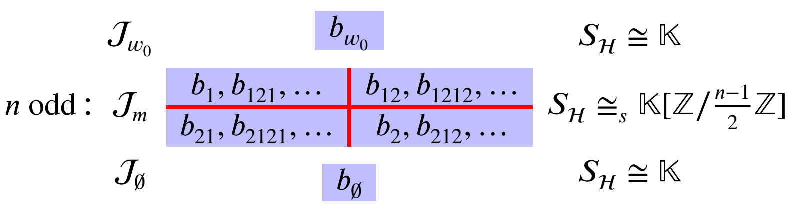


The Grothendieck algebra (abelian or additive, that does not make a difference) of $SO_3(\mathbb{C})$ can be computed via the SageMath online calculator, see above, with the code

```
A=WeylCharacterRing(A1,style=coroots);
k=5;
j=4;
A(2*k,0)*A(2*j,0)
```

You need to vary k and j, and identify b_{121} with $A(2) = A(2,0)$ up to scaling. Neither b_{121} nor $A(2)$ satisfy any polynomial relation, but both generate the respective algebras.

Hints for Exercise 4
Unless the characteristic of $\mathbb{K}$ is nonzero and small, the picture for n being odd should look like



That the diagonal $S_{\mathcal{H}}$ have pseudo idempotents is clear by $b_1 b_1 = 2b_1$. For the off-diagonal elements let us take $n = 7$ and $b = b_{12} - b_{1212} + b_{121212}$. Then the multiplication table

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	b_{12}	$-b_{1212}$	b_{121212}
b_{12}	$2b_{12} + b_{1212}$	$-b_{12} - 2b_{1212} - b_{121212}$	$b_{1212} + b_{121212}$
$-b_{1212}$	$-b_{12} - 2b_{1212} - b_{121212}$	$2b_{12} + 2b_{1212} + b_{121212}$	$-b_{12} - b_{1212}$
b_{121212}	$b_{1212} + b_{121212}$	$-b_{12} - b_{1212}$	b_{12}

verifies that $b^2 = b$. The general case is similar. (Note that b would be an infinite alternating sum for $n = \infty$, and that is why the off-diagonal $S_{\mathcal{H}}$ do not have pseudo idempotents in the infinite case.)

The isomorphism $S_{\mathcal{H}} \cong_s \mathbb{K}[\mathbb{Z}/\frac{n-1}{2}\mathbb{Z}]$ for nonsilly $\mathbb{K}$ can be verified as follows. Let $U_k^3(X)$ be the (Chebyshev-like multiplication by quantum three) polynomial defined via $U_0^3(X) = 1$, $U_1^3(X) = X$ and

$$U_k^3(X) = (X-1)U_{k-1}^3(X) - U_{k-2}^3(X).$$

This polynomial is the defining polynomial for $\mathrm{SO}_3(\mathbb{C})$ in the sense that $U_k^3(X)$ corresponds to the highest weight summand in the tensor product $(X = \mathbb{C}^3)^{\otimes k}$. Here is some SageMath code:

```
A=WeylCharacterRing(A1,style=coroots);
gen=A(2,0);
k=7;
def U(n,x):
if n == 0:
return 1
elif n == 1:
return x
else:
return (x-1) * U(n-1,x) - U(n-2,x)
print(U(k,gen))
```

Now $U_m^3(b_{121}) = 0$ for $m = \frac{n-1}{2}$, so $S_{\mathcal{H}} \cong_s \mathbb{K}[X]/(U_m^3(X))$. Since $U_m^3(X)$ has distinct roots, we can then rescale $\mathbb{K}[X]/(U_m^3(X))$ to $\mathbb{K}[X]/(X^m - 1) \cong \mathbb{K}[\mathbb{Z}/m\mathbb{Z}]$.

That was the case of $\mathrm{SO}_3(\mathbb{C})$, so you need to argue why this implies the same for the KL basis of the finite dihedral group.