Exercises 3

1. Twisting super vector spaces

Let $\mathscr{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}$ be the $\mathbb{C}$-linear monoidal category with objects given by the elements of $\mathbb{Z}/2\mathbb{Z}$, $\otimes$ on objects = group multiplication and trivial hom spaces.

Let $\omega\colon \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\to\mathbb{C}^{*}$ be the 3-cocycle with $\omega(1,1,1)=-1$, and twist $\mathscr{V}\mathrm{ec}^{1}_{\mathbb{Z}/2\mathbb{Z}}=\mathscr{V}\mathrm{ec}_{\mathbb{Z}/2\mathbb{Z}}$ on the associator α and unitors λ, ρ by

$$\alpha_{i,j,k}=\omega(i,j,k)\mathrm{id}_{ijk},\quad \lambda_i=\omega(1,1,i)^{-1}\mathrm{id}_i,\quad \rho_i=\omega(i,1,1)\mathrm{id}_i,$$

and obtain $\mathscr{V}\mathrm{ec}^{\omega}_{\mathbb{Z}/2\mathbb{Z}}$.

a) Show that $\mathscr{V}\mathrm{ec}^{\omega}_{\mathbb{Z}/2\mathbb{Z}}$ is nonstrict and skeletal at the same time.
b) Show that the additive Grothendieck algebra $[\_]_{\oplus}$ of the additive closure of $\mathscr{V}\mathrm{ec}^{x}_{\mathbb{Z}/2\mathbb{Z}}$ is $[\mathscr{V}\mathrm{ec}^{x}_{\mathbb{Z}/2\mathbb{Z}}]_{\oplus}\cong\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$, regardless of $x\in\{1,\omega\}$.
c) Show that $\mathscr{V}\mathrm{ec}^{1}_{\mathbb{Z}/2\mathbb{Z}}$ is not monoidally equivalent to $\mathscr{V}\mathrm{ec}^{\omega}_{\mathbb{Z}/2\mathbb{Z}}$.
d) (’) Classify braiding on $\mathscr{V}\mathrm{ec}^{x}_{\mathbb{Z}/2\mathbb{Z}}$ for $x\in\{1,\omega\}$ (up to braided equivalence).

2. Jordan decomposition in prime characteristic

Let $G=\mathbb{Z}/p\mathbb{Z}$ for some prime p, and let $\mathbb{K}=\overline{\mathbb{F}_p}$.

a) Show that $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$ is a fiat monoidal category with one simple and p indecomposable objects.
b) Compute the cell structure of $\mathscr{R}\mathrm{ep}(G,\mathbb{K})$.

3. Taft and friends

For fixed $n\in\mathbb{Z}_{>1}$, let $\mathscr{S}$ be a fiat monoidal category over $\mathbb{C}$ with indecomposable objects $Z_{l,r}$ for $1\leq l\leq n$ and $0\leq r\leq n-1$, with r read modulo n. The objects $L_r=Z_{1,r}$ are simple and their projective covers are $P_r=Z_{n,r}$.

The monoidal structure is $M\otimes N\cong N\otimes M$ and

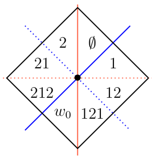
$$L_r\otimes L_{r'}\cong L_{r+r'},\quad L_r\otimes Z_{l,r'}\cong Z_{l,r+r'},\quad L_r\otimes P_{r'}\cong P_{r+r'},$$
$$Z_{l,r}\otimes Z_{l',r'}\cong\begin{cases}\bigoplus_{i=1}^{\min(l,l')}Z_{|l-l'|+2i-1,r+r'-i+\min(l,l')}, & \text{if } l+l'\leq n,\\ \bigoplus_{i=1}^{n-\max(l,l')}Z_{|l-l'|+2i-1,r+r'-i+\min(l,l')} & \\ \quad\oplus\bigoplus_{i=1}^{l+l'-n}P_{r+r'-i+1}, & \text{if } l+l'>n,\end{cases}$$
$$Z_{l,r}\otimes P_{r'}\cong\bigoplus_{i=1}^{l}P_{r+r'-i+l},\quad P_r\otimes P_{r'}\cong\bigoplus_{i=1}^{n}P_{r+r'-i+n}.$$

Let us define a two-variable Chebyshev-type polynomial by $U_0(X,Y)=1$, $U_1(X,Y)=X$ and $U_{e+1}(X,Y)=XU_e(X,Y)-YU_{e-1}(X,Y)$ for $e>0$. Let $[\_]_{exact}$ denote the abelian and $[\_]_{\oplus}$ the additive Grothendieck algebra.

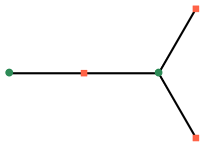
a) Compare $\mathscr{S}$ with $\mathscr{R}\mathrm{ep}(\mathbb{Z}/n\mathbb{Z},\mathbb{C})$ and $\mathscr{R}\mathrm{ep}(\mathbb{Z}/n\mathbb{Z},\overline{\mathbb{F}_n})$ (the latter assuming that n is prime).
b) Show that $[\mathscr{S}]_{exact}\cong\mathbb{Z}[Y]/(Y^n-1)$ but $[\mathscr{S}]_{\oplus}\cong\mathbb{Z}[X,Y]/(Y^n-1,(X-Y-1)U_{n-1}(X,Y))$.
c) Compute the cell structure of $\mathscr{S}$.

4. More fun with dihedral groups – $\mathbb{N}$-representations

As on the previous exercise sheet, let $\emptyset$ denote the unit and let $D_n = \langle 1,2 | 1^2 = 2^2 = (12)^n = \emptyset \rangle$ be the dihedral group of the n gon.

$D_4 \rightsquigarrow$ 

We also allow $n = \infty$, and we consider the KL basis $\{b_w | w \in D_n\}$ as before. We think of 1 as being colored spinach and 2 as being colored tomato. Take any simple connected bipartite graph $\Gamma = (V, E)$ with spinach $\underline{i}$ and tomato $\overline{i}$ colored vertices such as

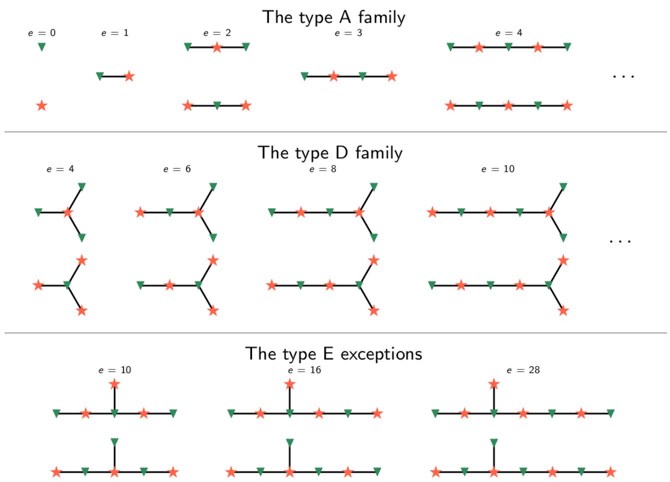
$\Gamma =$ 

Define an $\mathbb{N}$-representation of D_n on $\mathbb{N}V$ by

$$b_1 \cdot \underline{i} = 2 \cdot \underline{i}, \quad b_1 \cdot \overline{i} = \textstyle\sum_{\underline{j}-\overline{i}} \underline{j}, \quad b_2 \cdot \overline{i} = 2 \cdot \overline{i}, \quad b_2 \cdot \underline{i} = \textstyle\sum_{\overline{j}-\underline{i}} \overline{j},$$

where $a - b$ means a and b are connected.

a) Verify that the above defines an $\mathbb{N}$-representation of D_∞ for any graph Γ.
b) (*) Let $e = n - 2$. Verify that the above defines an $\mathbb{N}$-representation of D_n if and only if Γ is of ADE type



where type A_m shows up for $n = m + 1$, type D_m for $n = 2m - 2$ and types E_6, E_7 and E_8 for $n = 12, 18, 30$, respectively.
(Aside: $e = n - 2$ is often called the level, with 2 being a reference to $SL_2(\mathbb{C})$ or $SO_3(\mathbb{C})$ and the above is essentially a statement about $\mathbb{N}$-representations of the Grothendieck algebras of these beasts.)
c) (') If you know Soergel bimodules, then you should be able to guess a categorification.

- There might be typos on the exercise sheets, my bad. Be prepared.
- Star exercises are a bit trickier; prime exercises use notions I haven't explained.