Lecture 2: main exercises

Exercise 2.1. Recall from the lecture that

$$X(\mathbf{i}) \cong \{(z_1, \ldots, z_r) \in \mathbb{C}^r : B_{i_1}(z_1) \cdots B_{i_r}(z_r) \text{ has zeros above the antidiagonal}\}.$$

For $n = 2$, find the equations in z_1, z_2, z_3 cutting out $X(1,1,1)$. Draw the real points of $X(1,1,1)$ (i.e. the real solutions to these equations) in $\mathbb{R}^2$.

Exercise 2.2. Determine the values of z_1', z_2', z_3' that make the following equality hold:

$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_1')B_i(z_2')B_{i+1}(z_3').$$

Use this to re-derive 1.3(b): if $|i - j| = 1$, $\mathbf{i} = \mathbf{i}_1 iji \mathbf{i}_2$ and $\mathbf{j} = \mathbf{i}_1 jij \mathbf{i}_2$ then the varieties $X(\mathbf{i})$ and $X(\mathbf{j})$ are isomorphic.

Lecture 2: additional exercises

Exercise 2.3. Prove that if $\mathbf{i}$ is a reduced expression for w_0, then $X(\mathbf{i})$ is a point.

Exercise 2.4. Suppose $V \in Gr(2,n)$ is the column span of a $n \times 2$ matrix A. Write A_i for the ith row of A, and for $1 \leq i < j \leq n$, define

$$p_{ij}(A) := \det \begin{bmatrix} -A_i- \\ -A_j- \end{bmatrix}.$$

(a) Verify that if you choose another $n \times 2$ matrix B whose column span is V, then there is a constant $c \neq 0$ such that for all $1 \leq i < j \leq n$, $c \cdot p_{ij}(A) = p_{ij}(B)$. Use this to conclude that the map

$$\begin{aligned} \alpha : Gr(2,n) &\to \mathbb{P}^{\binom{n}{2}-1} \\ V &\mapsto \{p_{ij}(A) : 1 \leq i < j \leq n\} \end{aligned}$$

is well-defined.

(b) The map α is called the *Plücker embedding* of the Grassmannian, and $p_{ij}(A)$ is called a *Plücker coordinate* of V (and is usually denoted $p_{ij}(V)$). Show that α is injective, or equivalently that V is uniquely determined by its Plücker coordinates.

Hint: If A has an identity matrix in rows 1 and 2, how are the Plücker coordinates related to the entries of A?

(c) Verify (by computer if desired) that for $1 \leq i < j < k < \ell \leq n$, the following relation holds among the Plücker coordinates of $V \in Gr(2,n)$

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk}.$$

It turns out that these relations exactly describe the image of the Plücker embedding in $\mathbb{P}^{\binom{n}{2}-1}$.

(d) Let $w = (n-1)n12\cdots(n-2)$. Show that the open positroid variety Π_w^e is the subset of $Gr(2,n)$ where the Plücker coordinates $p_{12}, p_{23}, \ldots, p_{(n-1)n}, p_{1n}$ are nonvanishing.