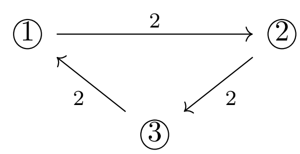
Exercise 3.1. Consider the quiver $\textcircled{1} \to \boxed{2}$ where the vertex 1 is mutable and the vertex 2 is frozen.

(a) Prove that the corresponding cluster algebra is of finite type, that is, there are finitely many seeds. Describe all the seeds and all the cluster variables in them.

(b) Describe the cluster algebra by generators and relations.

(c) Describe the corresponding cluster variety and prove that it is isomorphic to the braid variety $X(1,1,1)$.

Exercise 3.2. Consider the *Markov quiver*:



In the problems below you can use without proof that there are infinitely many seeds in the corresponding cluster algebra $\mathcal{A}$.

(a) Prove that this quiver stays the same under mutations. (The cluster variables do change though!)

(b) Prove that $\mathcal{A}$ is non-negatively graded with all cluster variables (in all seeds) of degree 1.

(c) Use (b) to prove that $\mathcal{A}$ is non-Noetherian and not finitely generated.

(d) Suppose that x, y, z are cluster variables in some seed and x' is obtained from x by mutation. Prove that

$$\frac{x^2+y^2+z^2}{xyz} = \frac{(x')^2+y^2+z^2}{x'yz}.$$

(e) Use part (d) to prove that the Laurent polynomial $m = \frac{x^2+y^2+z^2}{xyz}$ belongs to the upper cluster algebra $\mathcal{U}$. Prove that is does not belong to $\mathcal{A}$ and hence $\mathcal{A} \neq \mathcal{U}$. Hint: what is the degree of m?

Lecture 3: additional exercises

Exercise 3.3. Given a quiver with m mutable and f frozen vertices, consider the $m \times (m+f)$ matrix $B = (b_{ij})$ where b_{ij} is the (signed) number of arrows from a mutable vertex i to a vertex j. Prove that the rank of B does not change under mutations.

Exercise 3.4. Continuing with the Markov quiver, suppose that x_s, y_s, z_s are cluster variables in some seed s written as Laurent polynomials in the initial cluster variables x, y, z. Prove that the specializations of x_s, y_s, z_s at $x = y = z = 1$ give an integer solution of the Markov equation

$$a^2 + b^2 + c^2 = 3abc.$$