Lecture 4: main exercises

Exercise 4.1. Let $Q(A_n)$ be the following quiver, that has no frozen vertices:

$$\textcircled{1} \longrightarrow \textcircled{2} \longrightarrow \cdots \longrightarrow \textcircled{n}.$$

Show that $Q(A_n)$ has really full rank if and only if n is even.

Exercise 4.2. Let Q be a quiver, and let x be a frozen vertex of Q. Construct a new quiver Q' using the following procedure.

(a) Make the vertex x mutable.

(b) Add an arbitrary number of arrows (in any direction, as long as you don't create oriented 2-cycles) between x and frozen vertices of Q that did not previously have arrows to/from x.

(c) Add a new frozen vertex, y, and an arrow $x \to y$ (and this is the only arrow incident to y).

Show that if Q has really full rank, then so does Q'.

Exercise 4.3. Let Q be the Markov quiver introduced in the previous exercise sheet, with upper cluster algebra $\mathcal{U}$ and cluster algebra $\mathcal{A}$. The variables $x_1, x_2, x_3 \in \mathcal{A}$ are pairwise coprime, and they are coprime with x_1', x_2', x_3'. But $\mathcal{U} \not\subseteq \mathcal{A}$. Does this contradict the Starfish Lemma?

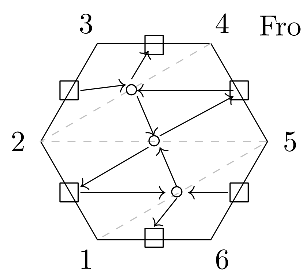
Lecture 4: additional exercises

Exercise 4.4. Let Q be an ice quiver. Show that if Q has really full rank, then any of its mutations also has really full rank.

Exercise 4.5. In this exercise we will construct a cluster structure on the coordinate ring of the positroid variety $\Pi_w^e \subseteq Gr(2,n)$, where $w = (n-1)n12\cdots(n-2)$. Recall that this is described as the set where the Plücker coordinates $p_{12}, p_{23}, \ldots, p_{(n-1)n)}, p_{1n}$ are nonvanishing.

Consider a convex n-gon P_n and let T be a triangulation by diagonals. To T, we associate a seed as follows:

- *Frozen variables*: $p_{12}, p_{23}, \ldots, p_{(n-1)n)}, p_{1n}$. These correspond to the sides of P_n.
- *Mutable variables*: p_{ij}, where ij runs over all diagonals of T.
- *Quiver*: Mutable vertices are in correspondence with diagonals of T, and frozen vertices with the sides of T. In each triangle we draw a counterclockwise cycle, ignoring arrows between frozen vertices. See the following example.



Frozen variables: $p_{12}, p_{23}, \ldots, p_{56}, p_{61}$

Mutable variables: p_{24}, p_{25}, p_{15}

(a) Consider a mutable variable corresponding to the diagonal ij. Note that this diagonal belongs to exactly two triangles, that together form a quadrilateral. Show that mutation at this variable corresponds to substituting this diagonal by the other diagonal in the same quadrilateral (in the example above, mutating at 25 substitutes 25 by 14).

(b) From the Starfish lemma, you may want to conclude that this gives a cluster structure on $\mathbb{C}[\Pi_w^e]$. But note that this cannot be true on the nose! In the example above, there are 9 cluster variables while $\dim \Pi_w^e = 2(6-2) = 8$. This can be explained from the fact that p_{ij} are *projective* coordinates. If p_{ij} is nowhere vanishing on Π_w^e, we may as well restrict to the chart where $p_{ij} \equiv 1$. Show that deleting one (any) frozen variable from the construction above gives a cluster structure on $\mathbb{C}[\Pi_w^e]$.

(c) Show that the cluster structure above has really full rank.