Lecture 6: main exercises

Exercise 6.1. Let $n = 2$ and $\beta = \sigma_1^k$. By definition, a point in $X(\beta)$ is a tuple of lines $(\ell_0, \ldots, \ell_k)$ in $\mathbb{C}^2$ such as $\ell_0 = \langle e_1 \rangle$, $\ell_k = \langle e_2 \rangle$ and $\ell_i \neq \ell_{i+1}$.

(a) Prove that Demazure weaves from β to $\delta(\beta) = \sigma_1$ are in bijection with the triangulations of the $(k+1)$-gon.

(b) Given an arbitrary weave for β, describe the corresponding torus in $X(\beta)$ by explicit inequalities between ℓ_i.

(c) Use (b) to prove that different weaves correspond to different tori in $X(\beta)$.

Exercise 6.2. Draw some weaves for

(a) $n = 3, \beta = 111222111$

(b) $n = 3, \beta = 11221122$ (compare with [CGGLSS,11.2])

(c) $n = 4, \beta = 213223122132$ (compare with [CGGLSS,11.4])

Exercise 6.3. (a) Suppose $z_2 \neq 0$. Prove that there exists an upper-triangular matrix U such that

$$B_i(z_1)B_i(z_2) = B_i(z_1 - z_2^{-1})U.$$

Hint: it is sufficient to work with 2×2 matrices

(b) Let U be some upper-triangular matrix and $z \in \mathbb{C}$. Prove that there exist unique $z' \in \mathbb{C}$ and an upper-triangular matrix U' such that

$$UB_i(z) = B_i(z')U'.$$

(c) Suppose $\beta = \ldots\sigma_i\sigma_i\ldots$ and $\beta' = \ldots\sigma_i\ldots$. Use parts (a) and (b) to construct an explicit map

$$X(\beta') \times \mathbb{C}^* \hookrightarrow X(\beta)$$

and describe its image in coordinates.

Exercise 6.4. Use Exercise 6.3 to prove that two equivalent weaves from $\sigma_1\sigma_2\sigma_1\sigma_2$ to $\sigma_2\sigma_1\sigma_2$ correspond to the same torus in $X(\beta)$.

Lecture 6: additional exercises

In the lecture it was stated that any two weaves built from 4- and 6-valent vertices are equivalent. This definition is sufficient for our purposes, and one need not delve into the details. However, there is a lot of interesting combinatorics in the details! These supplemental exercises explore what such weaves and equivalences look like. These weaves come from paths in a reduced expression graph.

The *reduced expression graph* of an element $w \in S_n$ is the graph defined as follows. Its vertices are reduced expressions for w. There is an edge between two reduced expressions if they are related by a single application of a braid relation. It helps to draw two different kinds of edges for the two different kinds of braid relations: a single edge for $s_i s_{i\pm1} s_i = s_{i\pm1} s_i s_{i\pm1}$, and a double edge for $s_i s_j = s_j s_i$.

Note: By Matsumoto's theorem, the reduced expression graph is connected. So you can construct it greedily: pick a reduced expression, and keep applying braid relations to it until you eventually find them all.

Exercise 6.5. (a) Draw the reduced expression graphs for $s_2s_1s_2s_3s_2$ and $s_1s_3s_2s_1s_3$ in S_4. One of these graphs should contain a square, which is called a *disjoint square* because it comes from applying braid relations to disjoint subwords.

(b) Draw the reduced expression graph for $s_1s_3s_5 \in S_6$. This graph contains a more interesting cycle.

(c) Draw the reduced expression graph for $s_1s_2s_1s_4 \in S_5$.

(d) The big one: draw the reduced expression graph for the longest element $s_1s_2s_1s_3s_2s_1 \in S_4$. In addition to some disjoint squares there should be one big cycle.

For answers, see `https://arxiv.org/pdf/1309.0865` pages 35-36.

Any vertex in a reduced expression graph corresponds to an expression, which is a horizontal cross-section of a weave. Any edge/braid relation corresponds to a 6-valent or 4-valent vertex; by putting the identity weave on the remainder of the expression, this edge yields a weave between the two expressions. So to any path in the reduced expression graph one can draw a weave using only 6-valent and 4-valent vertices.

Exercise 6.6. (a) Consider two paths from $s_1s_3s_2s_1s_3$ to $s_3s_1s_2s_3s_1$ along the two sides of the disjoint square. Draw the corresponding weaves. What do you notice? Does this hold true for disjoint squares in general?

(b) Draw weaves associated to the two distinct shortest paths from $s_1s_3s_5$ to $s_5s_3s_1$.

(c) Draw weaves associated to the two distinct shortest paths from $s_1s_2s_1s_4$ to $s_4s_2s_1s_2$.

(d) Draw weaves associated to two different paths from $s_1s_2s_1s_3s_2s_1$ to $s_3s_2s_3s_1s_2s_3$, going on opposite sides of the big cycle in this graph.

You answers should be similar to `https://arxiv.org/pdf/1309.0865` page 39.

According to the definition of weave equivalence, each pair of weaves you drew above (for part (a), for part (b), etc) is equivalent.

It turns out that, for any n, and for any $w \in S_n$, any two paths between two vertices in the reduced expression graph are related by a sequence of the equivalences you drew above (as well as boring equivalences which state that the identity is the same as going back and forth along an edge)! The cycles in any reduced expression graph are generated by disjoint squares and the cycles associated to the longest elements of rank 3 parabolic subgroups (types $A_1 \times A_1 \times A_1$, $A_1 \times A_2$, and A_3), which are sometimes called Zamolodchikov cycles. See `https://arxiv.org/pdf/1405.4928` for the reason why. One can similarly classify cycles in graphs of non-reduced expressions, see `https://arxiv.org/pdf/1907.10571`.

Executive summmary of `https://arxiv.org/pdf/1405.4928`: there is a contractible space built as a CW complex called the (completed dual) Coxeter complex. An expression corresponds to a path in the 1-skeleton. Braid relations correspond to 2-cells. A path in the reduced expression graph corresponds to a disk D^2 mapping to the 2-skeleton, whose boundary consists of the two paths in the 1-skeleton. (Weaves are actually drawings of this disk!) Zamolodchikov cycles correspond to the 3-skeleton. Since π_2 of a contractible space is trivial, and agrees with π_2 of the 3-skeleton, any two disks with the same boundary are related by homotopies within the 2-skeleton (boring equivalences, disjoint squares) or along 3-cells (Zamolodchikov equivalences).