Lecture 1: main exercises

Exercise 1.1. Consider a matrix $M \in \mathrm{GL}(n)$ and let $i = 1, \ldots, n-1$.

(a) Show that $\mathcal{F}_\bullet(M) \xrightarrow{s_i} \mathcal{F}'_\bullet$ if and only if there exists a unique $z \in \mathbb{C}$ such that

$$\mathcal{F}'_\bullet = \mathcal{F}_\bullet(MB_i(z)), \qquad B_i(z) = \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & z & -1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}$$

where the non-identity part of $B_i(z)$ is located at the i-th and $(i+1)$-st row and columns.

Solution: Suppose $\mathcal{F}_\bullet(M) \xrightarrow{s_i} \mathcal{F}'_\bullet$. Note that $\langle m_i \rangle \neq \mathcal{F}'_i/\mathcal{F}'_{i-1} \subseteq \mathcal{F}'_{i+1}/\mathcal{F}'_{i-1} = \mathcal{F}_{i+1}/\mathcal{F}_{i-1} = \langle m_i, m_{i+1} \rangle$. Then $\mathcal{F}'_i/\mathcal{F}'_{i-1}$ is a one-dimensional space spanned by $am_i + bm_{i+1}$ for $b \neq 0$, hence by $zm_i + m_{i+1}$ for $z = \frac{a}{b} \in \mathbb{C}$. So we have $\mathcal{F}'_\bullet = \mathcal{F}_\bullet(MB_i(z))$ for this choice of z.

For the other direction, note that $MB_i(z)$ has i^{th} column $zm_i + m_{i+1}$ and $(i+1)^{\text{st}}$ column $-m_{i+1}$, with all other columns equal to those of M. Since $zm_i + m_{i+1}$ is contained in $\mathcal{F}_{i+1}$ but not $\mathcal{F}_i$, we can see $\mathcal{F}'_i \neq \mathcal{F}_i$ and $\mathcal{F}'_j = \mathcal{F}_j$ for $j \neq i$. ■

(b) Thanks to part (a), if we fix a flag $\mathcal{F}_\bullet$ we have $\{\mathcal{F}'_\bullet \mid \mathcal{F}_\bullet \xrightarrow{s_i} \mathcal{F}'_\bullet\} \cong \mathbb{C}$. However, this parametrization depends on the matrix chosen to represent $\mathcal{F}_\bullet$: if $\mathcal{F}_\bullet(M) = \mathcal{F}_\bullet(M')$ then for every $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ such that $\mathcal{F}_\bullet(MB_i(z)) = \mathcal{F}_\bullet(M'B_i(w))$. How are M and M' related? How are z and w related?

Solution: From lecture, we have that $M' = MU$ for some upper triangular $U \in B$. Let a be the $(i, i+1)$-entry of U. By performing the corresponding row/column operations, we can check that $B_i^{-1}(a+z)UB_i(z)$ is upper-triangular. Thus $(M'B_i(a+z))^{-1}MB_i(z) = (MUB_i(a+z))^{-1}MB_i(z) = B_i^{-1}(a+z)UM^{-1}MB_i(z) = B_i^{-1}(a+z)UB_i(z) \in B$. Therefore, $\mathcal{F}_\bullet(MB_i(z)) = \mathcal{F}_\bullet(M'B_i(w))$ for $w = z + a$. ■

Exercise 1.2. (a) Let $w \in S_n$, assume $w(i) < w(i+1)$, so $w < ws_i$ in Bruhat order. Show that $(BwB)(Bs_iB) = Bws_iB$. Hint: What does a general matrix in wBs_i look like? It may be helpful to first think what a matrix in wB looks like

Solution: Consider the position of the pivots in the matrix representations of M_w of C_w and M_{s_i} of C_{s_i}. By assumption, the pivot in column i of M_w is above that of column $i+1$. When multiplying M_w on the right by M_{s_i}, we swap the position of the pivots in the two rows while all other entries to the left/above pivots remain free. Hence the result can be any matrix representative of C_{ws_i}. ■

(b) Under the assumptions of (a), conclude that:

- If $\mathcal{F}^1_\bullet, \mathcal{F}^2_\bullet, \mathcal{F}^3_\bullet$ are flags such that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$ and $\mathcal{F}^2_\bullet \xrightarrow{s_i} \mathcal{F}^3_\bullet$, then $\mathcal{F}^1_\bullet \xrightarrow{ws_i} \mathcal{F}^3_\bullet$.
  Solution: The assumptions are equivalent to $M_1^{-1}M_2 \in BwB$ and $M_2^{-1}M_3 \in Bs_iB$. Hence $M_1^{-1}M_3 = (M_1^{-1}M_2)(M_2^{-1}M_3) \in (BwB)(Bs_iB) = Bws_iB$. ■

- If $\mathcal{F}^1_\bullet \xrightarrow{ws_i} \mathcal{F}^3_\bullet$, then there exists a unique flag $\mathcal{F}^2_\bullet$ such that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$ and $\mathcal{F}^2_\bullet \xrightarrow{s_i} \mathcal{F}^3_\bullet$.
Solution: By assumption, $M_1^{-1}M_3 \in Bws_iB = (BwB)(Bs_iB)$. So there exists $U_1, U_2, U_3, U_4 \in B$ such that $M_1^{-1}M_3 = U_1wU_2U_3s_iU_4$. Set $M_2 = M_3(U_3s_iU_4)^{-1}$. Then $M_1^{-1}M_2 = U_1wU_2$ and $M_2^{-1}M_3 = U_3s_iU_4$. ■

Exercise 1.3. (a) Assume that we have $\mathcal{F}^1_\bullet \xrightarrow{s_i} \mathcal{F}^2_\bullet \xrightarrow{s_j} \mathcal{F}^3_\bullet \xrightarrow{s_i} \mathcal{F}^4_\bullet$, where $|i-j|=1$. Show that $\mathcal{F}^2_\bullet$ and $\mathcal{F}^3_\bullet$ are uniquely determined by $\mathcal{F}^1_\bullet$ and $\mathcal{F}^4_\bullet$.

Solution: First suppose $j = i+1$. In order to determine $\mathcal{F}^2_\bullet$ and $\mathcal{F}^3_\bullet$, it is enough to determine $\mathcal{F}^2_i$. Note that $\mathcal{F}^1_{i+1} \neq \mathcal{F}^4_{i+1}$. Since $\mathcal{F}^2_i \subseteq \mathcal{F}^1_{i+1} \cap \mathcal{F}^4_{i+1}$, we then have $\mathcal{F}^2_i = \mathcal{F}^1_{i+1} \cap \mathcal{F}^4_{i+1}$.

Next, suppose $i = j+1$. In this case, it is enough to determine $\mathcal{F}^2_{i+1}$. From the relations, we have $\mathcal{F}^2_{i+1} \supseteq \mathcal{F}^1_i \cup \mathcal{F}^4_i$. Since $\mathcal{F}^1_i \neq \mathcal{F}^4_i$, we can conclude $\mathcal{F}^2_{i+1} = \mathcal{F}^1_i + \mathcal{F}^4_i$. ■

(b) Conclude from a) that, if $\mathbf{i} = \mathbf{i}_1 iji \mathbf{i}_2$ and $\mathbf{j} = \mathbf{i}_1 jij \mathbf{i}_2$ then the varieties $X(\mathbf{i})$ and $X(\mathbf{j})$ are isomorphic.

Solution: Construct an isomorphism from $X(\mathbf{i})$ to $X(\mathbf{j})$ that preserves all flags except $\mathcal{F}^{|\mathbf{i}_1|+2}$ and $\mathcal{F}^{|\mathbf{i}_1|+3}$, which are mapped to the other unique choice of flags for the word $\mathbf{j}$ as determined in (a). ■

(c) Assume that we have $\mathcal{F}^1_\bullet \xrightarrow{s_i} \mathcal{F}^2_\bullet \xrightarrow{s_j} \mathcal{F}^3_\bullet$, where $|i-j|>1$. Show that $\mathcal{F}^2_\bullet$ is uniquely determined by $\mathcal{F}^1_\bullet$ and $\mathcal{F}^3_\bullet$.

Solution: In either case, we have $\mathcal{F}^2_i = \mathcal{F}^3_i$ and $\mathcal{F}^2_j = \mathcal{F}^1_j$, which determines $\mathcal{F}^2_\bullet$. ■ **Solution:** Additionally, $\mathcal{F}^2_k = \mathcal{F}^1_k = \mathcal{F}^3_k$ for $k \neq i, j$. ■

(d) Conclude from c) that if $\mathbf{i} = \mathbf{i}_1 ij \mathbf{i}_2$ and $\mathbf{j} = \mathbf{i}_1 ji \mathbf{i}_2$ then the varieties $X(\mathbf{i})$ and $X(\mathbf{j})$ are isomorphic.

Solution: Construct an isomorphism from $X(\mathbf{i})$ to $X(\mathbf{j})$ that preserves all flags except $\mathcal{F}^{|\mathbf{i}_1|+2}$, which is mapped to the other unique choice of flag for the word $\mathbf{j}$ as determined in (a). ■

Thanks to this, if $\mathbf{i}$ and $\mathbf{j}$ are words related by braid moves, then we have a canonical isomorphism $X(\mathbf{i}) \cong X(\mathbf{j})$. So $X(\mathbf{i})$ depends only on the braid $\beta = \sigma_{i_1} \cdots \sigma_{i_r} \in \mathrm{Br}_n$ defined by $\mathbf{i}$, and not on $\mathbf{i}$ itself. Hence the name *braid variety*.

Exercise 1.4. Let $\mathcal{F}^1_\bullet, \mathcal{F}^2_\bullet$ be flags in $\mathbb{C}^n$. In this exercise we will verify that there is a unique permutation w such that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$.

(a) Let $W_{ij} := (\mathcal{F}^1_i \cap \mathcal{F}^2_j)/(\mathcal{F}^1_i \cap \mathcal{F}^2_{j-1})$. Show that $\dim W_{ij}$ is 0 or 1.

Solution: Note that the projection $\varphi : W_{ij} \to \mathcal{F}^2_j/\mathcal{F}^2_{j-1}$ is actually an injection, since $[x] \in \ker(\varphi)$ if and only if $x \in \mathcal{F}^2_{j-1}$, which implies $[x] = [0]$ in the quotient space. Thus $\dim(W_{ij})$ is at most 1, since $\dim(\mathcal{F}^2_j/\mathcal{F}^2_{j-1}) = 1$. ■

(b) Let $w(j)$ be the minimum value of i such that $\dim W_{ij} = 1$. Show that w is a permutation.

Solution: Let $A_{i,j} = \mathcal{F}^1_i \cap \mathcal{F}^2_j$. Then $\dim W_{ij} = \dim A_{i,j} - \dim A_{i,j-1}$. Note that $\dim(W_{ij})$ is increasing in i. Hence we have

$$|\{j : w(j) \leq i\}| = \sum_{j=0}^{n} \dim(W_{ij}) = \dim(A_{i,n}) - \dim(A_{i,0}) = i\,.$$

Thus w is a permutation. ■

(c) Similarly, let $W'_{ij} = (\mathcal{F}^1_i \cap \mathcal{F}^2_j)/(\mathcal{F}^1_{i-1} \cap \mathcal{F}^2_j)$, and let $w'(i)$ be the minimum j such that $\dim W'_{ij} = 1$. Show that w' is the inverse of w.

Solution: Similar to what we did in the previous part, we have

$$|\{k \leq j : w(k) \leq i\}| = \sum_{k=0}^{j} \dim(W_{ij}) = \dim(A_{i,j})$$

and

$$|\{\ell \leq i : w'(\ell) \leq j\}| = \sum_{\ell=0}^{i} \dim(W'_{ij}) = \dim(A_{i,j})$$

Hence $w(j) = i \iff w'(i) = j$. ■

(d) Let e_i be a vector in $\mathcal{F}^1_i \cap \mathcal{F}^2_{w'(i)}$ that is *not* in $\mathcal{F}^1_{i-1}$. Show that $(e_1, \ldots, e_i)$ is a basis of $\mathcal{F}^1_i$ and $(e_{w(1)}, \ldots, e_{w(i)})$ is a basis of $\mathcal{F}^2_i$ for every i. Conclude that $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$.

Solution: The fact that $(e_1, \ldots, e_i)$ is a basis of $\mathcal{F}^1_i$ follows since each is not in the span of the previous ones. Using part (c) and the fact that $e_i \in \mathcal{F}^2_{w'(i)}$, we have

$$\mathcal{F}^2_j = \langle e_i : w'(i) \leq j \rangle = \langle e_{w(1)}, \ldots, e_{w(j)} \rangle\,.$$

Since $\dim \mathcal{F}^1_i \cap \mathcal{F}^2_j = |\{1, 2, \ldots, i\} \cap \{w(1), \ldots, w(j)\}|$, their relative position is as claimed. ■

Exercise 2.1. Recall from the lecture that

$$X(\mathbf{i}) \cong \{(z_1, \dots, z_r) \in \mathbb{C}^r : B_{i_1}(z_1) \cdots B_{i_r}(z_r) \text{ has zeros above the antidiagonal}\}.$$

For $n = 2$, find the equations in z_1, z_2, z_3 cutting out $X(1,1,1)$. Draw the real points of $X(1,1,1)$ (i.e. the real solutions to these equations) in $\mathbb{R}^2$.

Solution: Doing the matrix multiplication, we get $B_1(z_1)B_1(z_2)B_1(z_3) = \begin{bmatrix} z_1z_2z_3 - z_3 - z_1 & 1 - z_1z_2 \\ z_2z_3 - 1 & z_2 \end{bmatrix}$. Thus $X(1,1,1) = \{z_1, z_2, z_3 \in \mathbb{C}^3 : z_1 + z_3(1 - z_1z_2) = 0\}$. We can see that when $1 - z_1z_2 \neq 0$, then $z_3 = \frac{-z_1}{1 - z_1z_2}$. If $1 - z_1z_2 = 0$, then $z_1 = 0$, but this is not possible. So we have a unique choice for z_3 whenever $z_1z_2 \neq 1$. The set $\{z_1, z_2 \in \mathbb{R}^2 : z_1z_2 \neq 1\}$ cuts out a hyperbola. ■

Exercise 2.2. Determine the values of z_1', z_2', z_3' that make the following equality hold:

$$B_i(z_1)B_{i+1}(z_2)B_i(z_3) = B_{i+1}(z_1')B_i(z_2')B_{i+1}(z_3').$$

Use this to re-derive 1.3(b): if $|i - j| = 1$, $\mathbf{i} = \mathbf{i}_1 iji \mathbf{i}_2$ and $\mathbf{j} = \mathbf{i}_1 jij \mathbf{i}_2$ then the varieties $X(\mathbf{i})$ and $X(\mathbf{j})$ are isomorphic.

Solution: Note that it is enough to work in the 3×3 submatrix of these products that is not equal to the identity. Expanding the matrix products, we get

$$\begin{bmatrix} z_1z_3 - z_2 & -z_1 & 1 \\ z_3 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} z_2' & -z_3' & 1 \\ z_1' & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

from which we can see that we must have $z_3' = -z_1$, $z_1' = z_3$, and $z_2' = z_1z_3 - z_2$. Thus each set of parameters is uniquely determined by the other. ■

Exercise 2.3. Prove that if $\mathbf{i}$ is a reduced expression for w_0, then $X(\mathbf{i})$ is a point.

Solution: From lecture, we have $\dim X(\mathbf{i}) = \ell(\mathbf{i}) - \ell(\mathbf{w_0}) = \mathbf{0}$. ■

Exercise 2.4. Suppose $V \in Gr(2,n)$ is the column span of a $n \times 2$ matrix A. Write A_i for the ith row of A, and for $1 \leq i < j \leq n$, define

$$p_{ij}(A) := \det \begin{bmatrix} -A_i- \\ -A_j- \end{bmatrix}.$$

(a) Verify that if you choose another $n \times 2$ matrix B whose column span is V, then there is a constant $c \neq 0$ such that for all $1 \leq i < j \leq n$, $c \cdot p_{ij}(A) = p_{ij}(B)$. Use this to conclude that the map

$$\alpha : Gr(2,n) \to \mathbb{P}^{\binom{n}{2}-1}$$
$$V \mapsto \{p_{ij}(A) : 1 \leq i < j \leq n\}$$

is well-defined.

Solution: There must be some $M \in \mathrm{GL}_2(\mathbb{R})$ such that $B = AM$. Then computing the minors, we see that we have $c = \det(M)$. So projectivizing to remove the constant multiplier, we get a well-defined map. ■

(b) The map α is called the *Plücker embedding* of the Grassmannian, and $p_{ij}(A)$ is called a *Plücker coordinate* of V (and is usually denoted $p_{ij}(V)$). Show that α is injective, or equivalently that V is uniquely determined by its Plücker coordinates.

Hint: If A has an identity matrix in rows 1 and 2, how are the Plücker coordinates related to the entries of A?

Solution: Let $B = \mathrm{RREF}(A) = AM$ for $m \in \mathrm{GL}_2(\mathbb{R})$. Then the entries of B appear as Plucker coordinates, taking one of the rows to be the first or second. We can recover the entries of A by dividing by $p_{12}(A) = \det(M)$. ■

(c) Verify (by computer if desired) that for $1 \leq i < j < k < \ell \leq n$, the following relation holds among the Plücker coordinates of $V \in Gr(2,n)$

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk}.$$

It turns out that these relations exactly describe the image of the Plücker embedding in $\mathbb{P}^{\binom{n}{2}-1}$.

Solution: Just expand in terms of entries of A, check the algebra works out. ■

(d) Let $w = (n-1)n12\cdots(n-2)$. Show that the open positroid variety Π_w^e is the subset of $Gr(2,n)$ where the Plücker coordinates $p_{12}, p_{23}, \ldots, p_{(n-1)n}, p_{1n}$ are nonvanishing.

Solution: First think of the shape of matrices in C_w. These look like

$$\begin{bmatrix} * & * & 1 & 0 & 0 & \cdots \\ * & * & 0 & 1 & 0 & \cdots \\ * & * & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

Then intersecting with C^e means we are enforcing $\Delta_{[i],[i]} \neq 0$. Each such minor is equal to one of the consecutive Plücker coordinates in the first two columns, namely $p_{i-1,i}$. ■

Exercise 3.1. Consider the quiver ① → [2] where the vertex 1 is mutable and the vertex 2 is frozen.

(a) Prove that the corresponding cluster algebra is of finite type, that is, there are finitely many seeds. Describe all the seeds and all the cluster variables in them.

Solution: By performing 2 mutations, we first get the cluster (x_1', x_2) with $x_1' = \frac{1+x_2}{x_1}$ and the arrow in the quiver reversed. The next mutation brings us back to the original quiver, and we also get the cluster (x_1'', x_2) with $x_1'' = \frac{1+x_2}{x_1'} = x_1$. Thus this is equivalent to the initial seed and we have two distinct seeds. ■

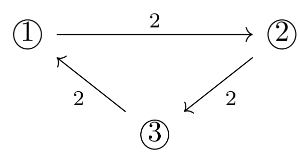
(b) Describe the cluster algebra by generators and relations.

Solution: $\mathcal{A} = \{x, x', y^{\pm} \in \mathbb{C} : xx' = 1 + y\}$. ■

(c) Describe the corresponding cluster variety and prove that it is isomorphic to the braid variety $X(1, 1, 1)$.

Solution: Since y is invertible, we have $\operatorname{Spec}(\mathcal{A}) = \{x, x' \in \mathbb{C}^2 : xx' \neq 1\}$. This matches up with what we saw in Lecture 2, Exercise 1. ■

Exercise 3.2. Consider the *Markov quiver*:



In the problems below you can use without proof that there are infinitely many seeds in the corresponding cluster algebra $\mathcal{A}$.

(a) Prove that this quiver stays the same under mutations. (The cluster variables do change though!)

Solution: Enough to perform a single mutation. ■

(b) Prove that $\mathcal{A}$ is non-negatively graded with all cluster variables (in all seeds) of degree 1.

Solution: Proceed by induction, noting that at each mutation we get $\frac{(\deg 1)^2 + (\deg 1)^2}{\deg 1} = \frac{\deg 2}{\deg 1} = \deg 1$. ■

(c) Use (b) to prove that $\mathcal{A}$ is non-Noetherian and not finitely generated.

Solution: Since $\mathcal{A}$ is non-negatively graded, it is finitely-generated only if $\mathcal{A}_1$ is. So we need to show there are infinitely many cluster variables that are not linearly related. To see this, consider alternating mutations at the two vertices 1 and 2 (with cluster variables x_n and y_n), and look at the degree in the thid cluster variable z. We can then check that the degree of x_n, y_n is strictly increasing in z, so these avoid linear relations. Hence $\mathcal{A}$ is not finitely generated. Letting $S_i = \{x_i, x_{i+1}, \dots\}$ and considering the chain $\mathcal{A}S_1 \subset \mathcal{A}S_2 \subset \cdots$ exhibits that it is non-Noetherian. ■

(d) Suppose that x, y, z are cluster variables in some seed and x' is obtained from x by mutation. Prove that

$$\frac{x^2 + y^2 + z^2}{xyz} = \frac{(x')^2 + y^2 + z^2}{x'yz}.$$

Solution: We have $x' = \frac{y^2+z^2}{x}$, so

$$\frac{\left(\frac{y^2+z^2}{x}\right)^2 + y^2 + z^2}{\frac{y^2+z^2}{x} yz} = \frac{(x^2 + y^2 + z^2)(y^2 + z^2)}{(y^2 + z^2)xyz} = \frac{x^2 + y^2 + z^2}{xyz}.$$

■

(e) Use part (d) to prove that the Laurent polynomial $m = \frac{x^2+y^2+z^2}{xyz}$ belongs to the upper cluster algebra $\mathcal{U}$. Prove that is does not belong to $\mathcal{A}$ and hence $\mathcal{A} \neq \mathcal{U}$. Hint: what is the degree of m?

Solution: From (d), it's a Laurent polynomial in every seed and hence in $\mathcal{U}$. But its degree is -1, so it is not in $\mathcal{A}$ which is nonnegatively graded by (b). ■

Exercise 3.3. Given a quiver with m mutable and f frozen vertices, consider the $m \times (m+f)$ matrix $B = (b_{ij})$ where b_{ij} is the (signed) number of arrows from a mutable vertex i to a vertex j. Prove that the rank of B does not change under mutations.

Solution: Mutation can be represented by multiplication on the left and right by two invertible matrices, L and R. Suppose we are mutating at k. We define $L \in \mathrm{GL}_m(\mathbb{Z})$ and $R \in \mathrm{GL}_{m+f}(\mathbb{Z})$ by

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \neq k \\ -1 & \text{if } i = j = k \\ [b_{ij}]_+ & \text{if } i \neq j = k \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad R_{ij} = \begin{cases} 1 & \text{if } i = j \neq k \\ -1 & \text{if } i = j = k \\ [-b_{ij}]_+ & \text{if } k = i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Both L and R are triangular except for one row/column and has entries ± 1 on the diagonal, so both have determinant ± 1. Hence this multiplication does not affect the rank of B. ■

Exercise 3.4. Continuing with the Markov quiver, suppose that x_s, y_s, z_s are cluster variables in some seed s written as Laurent polynomials in the initial cluster variables x, y, z. Prove that the specializations of x_s, y_s, z_s at $x = y = z = 1$ give an integer solution of the Markov equation

$$a^2 + b^2 + c^2 = 3abc.$$

Solution: By the Laurent phenomenon, all cluster variables are integers when we specialize the cluster variables to 1. We show the quantity $\frac{x^2+y^2+z^2}{xyz}$ is the same for all clusters, and for the initial cluster this quantity is 3. Thus for all clusters (x, y, z), we have $\frac{x^2+y^2+z^2}{xyz} = 3$, which is equivalent to the Markov equation. ■

Exercise 4.1. Let $Q(A_n)$ be the following quiver, that has no frozen vertices:

$$\textcircled{1} \longrightarrow \textcircled{2} \longrightarrow \cdots \longrightarrow \textcircled{n}.$$

Show that $Q(A_n)$ has really full rank if and only if n is even.

Solution: When n is even, taking row sums of the form r_1, $r_1 + r_3$, $r_1 + r_3 + r_5$, $\ldots$ recovers all the even basis vectors. Taking row sums of the form r_{2n}, $r_{2n} + r_{2n-2}$, $r_{2n} + r_{2n-2} + r_{2n-4}$, $\ldots$ yields the odd basis vectors.

When n is odd, note that the only rows a nonzero entries in the odd columns are the even rows. But there are more odd columns than even rows, so the rows cannot span the odd-indexed entries of $\mathbb{Z}^n$. ■

Exercise 4.2. Let Q be a quiver, and let x be a frozen vertex of Q. Construct a new quiver Q' using the following procedure.

(a) Make the vertex x mutable.

(b) Add an arbitrary number of arrows (in any direction, as long as you don't create oriented 2-cycles) between x and frozen vertices of Q that did not previously have arrows to/from x.

(c) Add a new frozen vertex, y, and an arrow $x \to y$ (and this is the only arrow incident to y).

Show that if Q has really full rank, then so does Q'.

Solution: By making x mutable, we are adding a column to B. Note that the other columns are unaffected, other than adding an additional 0 entry in the row for y. So these can be generated in $\mathbb{Z}$ as before. We can also generate the new copy of $\mathbb{Z}$ using the row for y, which is 1 in the row for x and 0 otherwise. ■

Exercise 4.3. Let Q be the Markov quiver introduced in the previous exercise sheet, with upper cluster algebra $\mathcal{U}$ and cluster algebra $\mathcal{A}$. The variables $x_1, x_2, x_3 \in \mathcal{A}$ are pairwise coprime, and they are coprime with x_1', x_2', x_3'. But $\mathcal{U} \not\subseteq \mathcal{A}$. Does this contradict the Starfish Lemma?

Solution: As we showed in an earlier exercise, $\mathcal{A}$ is not finitely generated as a $\mathbb{C}$-algebra, which is a hypothesis of the Starfish Lemma. ■

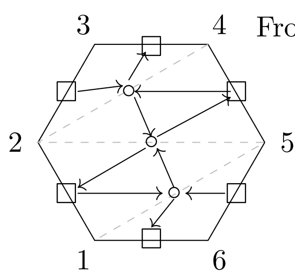
Exercise 4.4. Let Q be an ice quiver. Show that if Q has really full rank, then any of its mutations also has really full rank.

Solution: In the proof that mutation preserves full rank, we modeled mutation as multiplication on the left and right by two full-rank matrices. It it enough to check that these matrices have determinant ± 1. ■

Exercise 4.5. In this exercise we will construct a cluster structure on the coordinate ring of the positroid variety $\Pi_w^e \subseteq Gr(2, n)$, where $w = (n-1)n12\cdots(n-2)$. Recall that this is described as the set where the Plücker coordinates $p_{12}, p_{23}, \ldots, p_{(n-1)n)}, p_{1n}$ are nonvanishing.

Consider a convex n-gon P_n and let T be a triangulation by diagonals. To T, we associate a seed as follows:

- *Frozen variables*: $p_{12}, p_{23}, \ldots, p_{(n-1)n)}, p_{1n}$. These correspond to the sides of P_n.
- *Mutable variables*: p_{ij}, where ij runs over all diagonals of T.
- *Quiver*: Mutable vertices are in correspondence with diagonals of T, and frozen vertices with the sides of T. In each triangle we draw a counterclockwise cycle, ignoring arrows between frozen vertices. See the following example.



Frozen variables: $p_{12}, p_{23}, \ldots, p_{56}, p_{61}$

Mutable variables: p_{24}, p_{25}, p_{15}

(a) Consider a mutable variable corresponding to the diagonal ij. Note that this diagonal belongs to exactly two triangles, that together form a quadrilateral. Show that mutation at this variable corresponds to substituting this diagonal by the other diagonal in the same quadrilateral (in the example above, mutating at 25 substitutes 25 by 14).

Solution: Enough to check within a single quadrilateral, in which case you can directly compute the quivers. ■

(b) From the Starfish lemma, you may want to conclude that this gives a cluster structure on $\mathbb{C}[\Pi_w^e]$. But note that this cannot be true on the nose! In the example above, there are 9 cluster variables while $\dim \Pi_w^e = 2(6-2) = 8$. This can be explained from the fact that p_{ij} are *projective* coordinates. If p_{ij} is nowhere vanishing on Π_w^e, we may as well restrict to the chart where $p_{ij} \equiv 1$. Show that deleting one (any) frozen variable from the construction above gives a cluster structure on $\mathbb{C}[\Pi_w^e]$.

(c) Show that the cluster structure above has really full rank.

A quiver is called **isolated** if there are no arrows between mutable vertices.

Exercise 5.1. Let $\mathcal{A}$ be an isolated cluster algebra.

(a) Describe all cluster variables (in all clusters) in $\mathcal{A}$.

Solution: We get two cluster variables for each vertex, as we always have the relation $X_i X_i' = 1 + 1 = 2$. Mutation is an involution. So there are 2^m clusters, for each mutable choosing X_i or X_i'. Adding in frozens only changes the relations, but not the structure. ■

(b) Describe $\mathcal{A}$ by generators and relations.

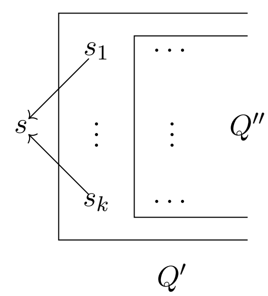
Solution:

$$\mathcal{A} = \langle X_i, X_i' : 1 \leq i \leq n, X_i X_i' = 2 \rangle .$$

■

The class of **sink-recurrent quivers** is defined recursively as follows. Assume first that Q has no frozen vertices.

- Any isolated quiver Q is sink-recurrent.
- Any quiver that is mutation equivalent to a sink-recurrent quiver is sink-recurrent.
- Suppose that a quiver Q has a sink vertex s, let $N(s) = \{s_1, \ldots, s_k\}$ be the set of neighbors of s. If the quivers $Q' = Q - \{s\}$ and $Q'' = Q - \{s\} - N(s)$ are sink-recurrent, then Q is sink-recurrent.



More generally, Q is sink-recurrent if its mutable part is sink-recurrent.

Exercise 5.2. (a) Assume that Q has a sink vertex s as above, and X is the corresponding cluster variety. Prove that

$$X = \{x_s \neq 0\} \cup \{x_{s_1} \cdots x_{s_k} \neq 0\}.$$

Solution: We want to show that in X, we cannot have both $x_s = 0$ and $x_{s_1} \cdots x_{s_k} = 0$ simultaneously. Look at the mutation relation for s. We have $x_s x_s' = x_{s_1} \cdots x_{s_k} + 1$, from which we can see x_s and $x_{s_1} \cdots x_{s_k}$ cannot both vanish. ■

(b) Assume that Q is sink-recurrent. Prove that the corresponding cluster algebra is locally acyclic.

Solution: Check operation by operation.

- no arrows between frozen, so any cycle involves two mutables (which don't occur if isolated)

- whether a quiver is locally acyclic is unaffected by mutation
- By part (a), $X = \{x_s \neq 0\} \cup \{x_n \neq 0 : x_n \in N(s)\}$. So it is enough to show that freezing s results in a locally acyclic quiver, and same for freezing $N(s)$. The latter differs from $Q - \{s\} - N(s)$ by an isolated vertex, so it's also locally acyclic.

■

Lecture 5: additional exercises

Exercise 5.3. Suppose Q is an acyclic quiver. Prove that every edge between mutable vertices in Q is separating.

Solution: There are no directed cycles in Q. Since there are finitely many vertices, any infinite directed path must return to itself, forming a directed cycle. ■

Exercise 5.4. (a) Prove that any acyclic cluster variety can be covered by isolated cluster charts. Hint: use Exercise 5.3

Solution: From lecture, we can freeze at each vertex on a separating edge to get cluster charts. Repeating this, we can freeze all except one vertex, and iterating over all choices gives isolated cluster charts. ■

(b) Prove that any locally acyclic cluster variety can be covered by isolated cluster charts.

Solution: Each locally acyclic cluster variety can be covered by finitely many acyclic charts. We can then apply (a). ■

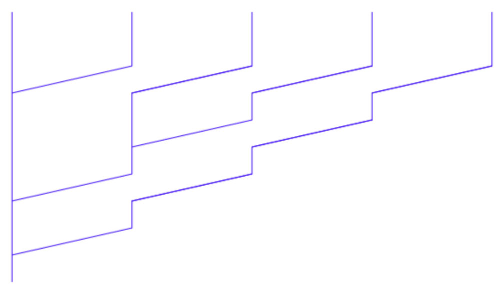
Exercise 6.1. Let $n = 2$ and $\beta = \sigma_1^k$. By definition, a point in $X(\beta)$ is a tuple of lines $(\ell_0, \ldots, \ell_k)$ in $\mathbb{C}^2$ such as ℓ_0 is standard, ℓ_k is antistandard and $\ell_i \neq \ell_{i+1}$.

(a) Prove that Demazure weaves from β to $\delta(\beta) = \sigma_1$ are in bijection with the triangulations of the $(k+1)$-gon.

Solution: Given a braid word $\beta = \sigma^k$, the Demazure weave has $(k-1)$trivalents which is in direct correspondence with the $(k-1)$-diagonals of a $(k+1)$-gon. ■

(b) Given an arbitrary weave for β, describe the corresponding torus in $X(\beta)$ by inequalities between ℓ_i.

Solution: Given an arbitrary weave for β, each of the $k+1$ chambers given at the top define a line ℓ_i. If two lines ℓ_i, ℓ_j share an edge of the weave then $\ell_i \neq \ell_j$ in the corresponding torus. For example, in the following weave the the correspoding torus in $X(\beta)$ is defined by the following inequalities: $\ell_0 \neq \ell_1, \ell_1 \neq \ell_2, \ell_2 \neq \ell_3, \ell_3 \neq \ell_4, \ell_4 \neq \ell_5, \ell_0 \neq \ell_2, \ell_0 \neq \ell_4, \ell_0 \neq \ell_5$.



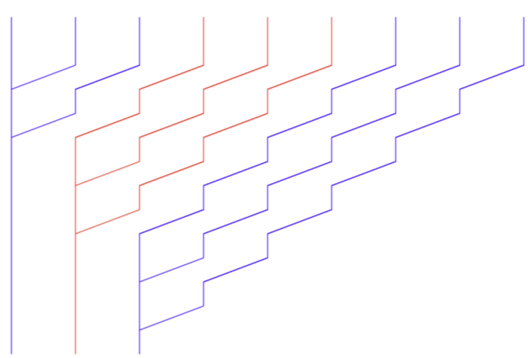
■

(c) Use (b) to prove that different weaves correspond to different tori in $X(\beta)$.

Exercise 6.2. Draw some weaves for

(a) $n = 3, \beta = 111222111$

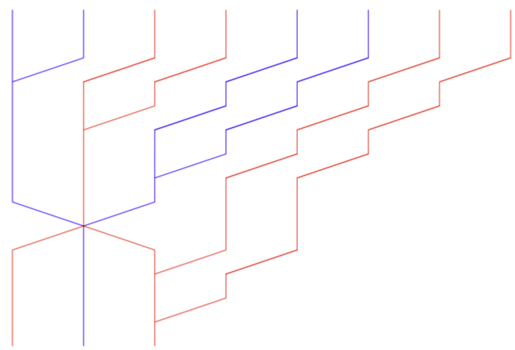
Solution: The right inductive weave:



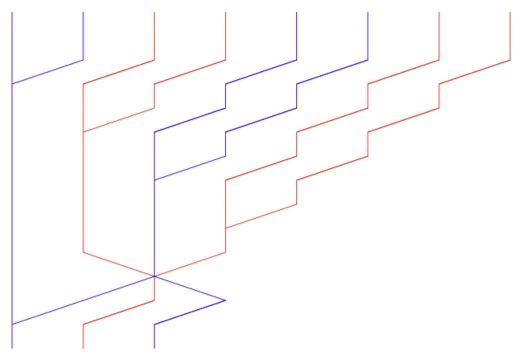
■

(b) $n = 3, \beta = 11221122$ (compare with [CGGLSS,11.2])

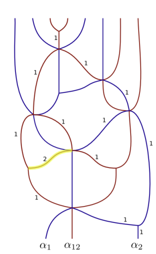
Solution: The right inductive weave:



An arbitrary weave:



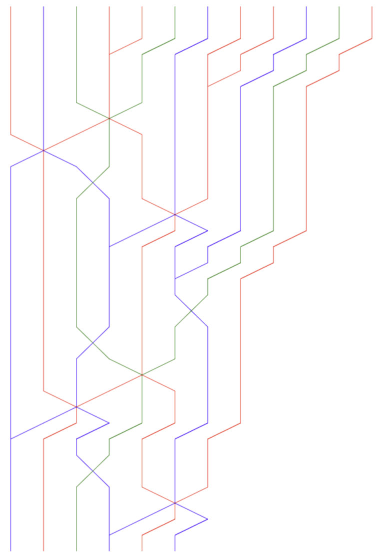
Compare these to [CGGLSS,11.2]:



■

(c) $n = 4, \beta = 213223122132$ (compare with [CGGLSS,11.4])

Solution: The right inductive weave [CGGLSS,11.4]:



■

Exercise 6.3. (a) Suppose $z_2 \neq 0$. Prove that there exists an upper-triangular matrix U such that

$$B_i(z_1)B_i(z_2) = B_i(z_1 - z_2^{-1})U.$$

Hint: it is sufficient to work with 2×2 matrices

Solution: We begin by multiplying the 2×2 matrices

$$B_1(z_1)B_1(z_2) = \begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z_1z_2 - 1 & z_1 \\ z_2 & -1 \end{pmatrix}$$

$$B_1(z_1 - z_2^{-1})U = \begin{pmatrix} z_1 - z_2^{-1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} az_1 - az_2^{-1} & bz_1 - bz_2^{-1} - c \\ a & b \end{pmatrix}$$

Solving for a, b, c, we find that $a = z_2$, $b = -1$, $c = z_2^{-1} - 2z_1$. ■

(b) Let U be some upper-triangular matrix and $z \in \mathbb{C}$. Prove that there exist unique $z' \in \mathbb{C}$ and an upper-triangular matrix U' such that

$$UB_i(z) = B_i(z')U'.$$

Solution: We begin by multiply the matrices

$$UB_1(z) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} az + b & -a \\ c & 0 \end{pmatrix}$$

$$B_1(z')U' = \begin{pmatrix} z' & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} a'z' & b'z' - c' \\ a' & b' \end{pmatrix}$$

Solving for a', b', c', z', we find that $a' = c$, $b' = 0$, $c' = a$, $z' = \frac{a}{c}z + \frac{b}{c}$. ■

(c) Suppose $\beta = \ldots\sigma_i\sigma_i\ldots$ and $\beta' = \ldots\sigma_i\ldots$. Use parts (a) and (b) to construct an explicit map

$$X(\beta') \times \mathbb{C}^* \hookrightarrow X(\beta)$$

and describe its image in coordinates.

Exercise 6.4. Use Exercise 6.3 to prove that two equivalent weaves from $\sigma_1\sigma_2\sigma_1\sigma_2$ to $\sigma_2\sigma_1\sigma_2$ correspond to the same torus in $X(\beta)$.

Lecture 6: additional exercises

In the lecture it was stated that any two weaves built from 4- and 6-valent vertices are equivalent. This definition is sufficient for our purposes, and one need not delve into the details. However, there is a lot of interesting combinatorics in the details! These supplemental exercises explore what such weaves and equivalences look like. These weaves come from paths in a reduced expression graph.

The *reduced expression graph* of an element $w \in S_n$ is the graph defined as follows. Its vertices are reduced expressions for w. There is an edge between two reduced expressions if they are related by a single application of a braid relation. It helps to draw two different kinds of edges for the two different kinds of braid relations: a single edge for $s_i s_{i\pm1} s_i = s_{i\pm1} s_i s_{i\pm1}$, and a double edge for $s_i s_j = s_j s_i$.

Note: By Matsumoto's theorem, the reduced expression graph is connected. So you can construct it greedily: pick a reduced expression, and keep applying braid relations to it until you eventually find them all.

Exercise 6.5. (a) Draw the reduced expression graphs for $s_2s_1s_2s_3s_2$ and $s_1s_3s_2s_1s_3$ in S_4. One of these graphs should contain a square, which is called a *disjoint square* because it comes from applying braid relations to disjoint subwords.

Solution: (a) 12312 = 12132 — 21232 — 21323 = 23123

(b)

13213 = 13231 — 12321
‖ ‖
32123 — 31213 = 31231

■

(b) Draw the reduced expression graph for $s_1s_3s_5 \in S_6$. This graph contains a more interesting cycle.

Solution:

315 = 351 = 531
‖ ‖
135 = 153 = 513

■

(c) Draw the reduced expression graph for $s_1s_2s_1s_4 \in S_5$.

Solution:

1214 — 2124 = 2142 = 2412
‖ ‖
1241 = 1421 = 4121 — 4212

■

(d) The big one: draw the reduced expression graph for the longest element $s_1s_2s_1s_3s_2s_1 \in S_4$. In addition to some disjoint squares there should be one big cycle.

For answers, see `https://arxiv.org/pdf/1309.0865` pages 35-36.

Any vertex in a reduced expression graph corresponds to an expression, which is a horizontal cross-section of a weave. Any edge/braid relation corresponds to a 6-valent or 4-valent vertex; by putting the identity weave on the remainder of the expression, this edge yields a weave between the two expressions. So to any path in the reduced expression graph one can draw a weave using only 6-valent and 4-valent vertices.

Exercise 6.6. (a) Consider two paths from $s_1s_3s_2s_1s_3$ to $s_3s_1s_2s_3s_1$ along the two sides of the disjoint square. Draw the corresponding weaves. What do you notice? Does this hold true for disjoint squares in general?

Solution: It's two 4-valent crossings on disjoint pairs of strings, just changing the order in which they occur. ■

(b) Draw weaves associated to the two distinct shortest paths from $s_1s_3s_5$ to $s_5s_3s_1$.

Solution: These consist only of 4-valent crossings (no longer on disjoint pairs of strings), just changing the order. ■

(c) Draw weaves associated to the two distinct shortest paths from $s_1s_2s_1s_4$ to $s_4s_2s_1s_2$.

Solution: One starts with a 6-valent crossing and then three 4-valent crossings. The other starts with three 4-valent crossings followed by a 6-valent crossing. ■

(d) Draw weaves associated to two different paths from $s_1s_2s_1s_3s_2s_1$ to $s_3s_2s_3s_1s_2s_3$, going on opposite sides of the big cycle in this graph.

You answers should be similar to `https://arxiv.org/pdf/1309.0865` page 39.

According to the definition of weave equivalence, each pair of weaves you drew above (for part (a), for part (b), etc) is equivalent.

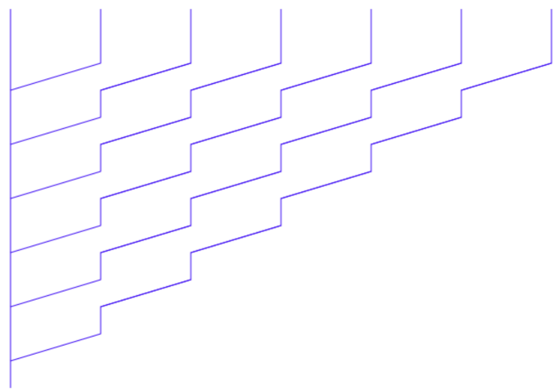
It turns out that, for any n, and for any $w \in S_n$, any two paths between two vertices in the reduced expression graph are related by a sequence of the equivalences you drew above (as well as boring equivalences which state that the identity is the same as going back and forth along an edge)! The cycles in any reduced expression graph are generated by disjoint squares and the cycles associated to the longest elements of rank 3 parabolic subgroups (types $A_1 \times A_1 \times A_1$, $A_1 \times A_2$, and A_3), which are sometimes called Zamolodchikov cycles. See `https://arxiv.org/pdf/1405.4928` for the reason why. One can similarly classify cycles in graphs of non-reduced expressions, see `https://arxiv.org/pdf/1907.10571`.

Executive summmary of `https://arxiv.org/pdf/1405.4928`: there is a contractible space built as a CW complex called the (completed dual) Coxeter complex. An expression corresponds to a path in the 1-skeleton. Braid relations correspond to 2-cells. A path in the reduced expression graph corresponds to a disk D^2 mapping to the 2-skeleton, whose boundary consists of the two paths in the 1-skeleton. (Weaves are actually drawings of this disk!) Zamolodchikov cycles correspond to the 3-skeleton. Since π_2 of a contractible space is trivial, and agrees with π_2 of the 3-skeleton, any two disks with the same boundary are related by homotopies within the 2-skeleton (boring equivalences, disjoint squares) or along 3-cells (Zamolodchikov equivalences).

Exercise 7.1. For each of the following braids:
(a) Draw its inductive weave.
(b) Find its solid crossings.
(c) Determine which Deodhar hypersurfaces are frozen, and which are mutable.

(i) $\beta = \sigma_1^n \in \mathrm{Br}_2^+$. For this braid, also find a recursive formula for the chamber minors.

Solution:

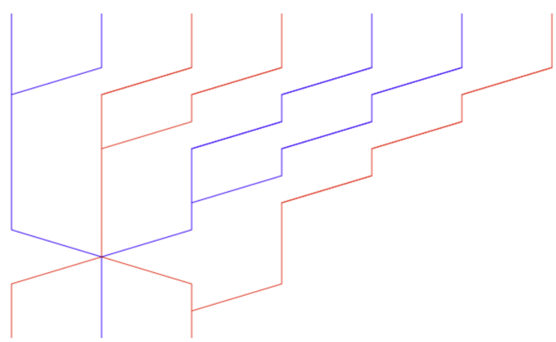


The solid crossings are all σ_1 after the initial σ_1, since the Demazure product $\delta(\sigma_1^n) = \sigma_1$. *Guess: The chamber minors are defined by the recursive definition* $F_n(z_1, \ldots, z_n) = z_n F_{n-1}(z_1, \ldots, z_{n-1}) - F_{n-2}(z_1, \ldots, z_{n-2})$ where $F_0 \equiv 0, F_1 \equiv 1, F_1 = z_1$?? ■

Solution: All the solid crossings except the last one are mutable, and the last is frozen ■

(ii) $\beta = \sigma_1^2 \sigma_2^2 \sigma_1^2 \sigma_2 \in \mathrm{Br}_3^+$.

Solution:

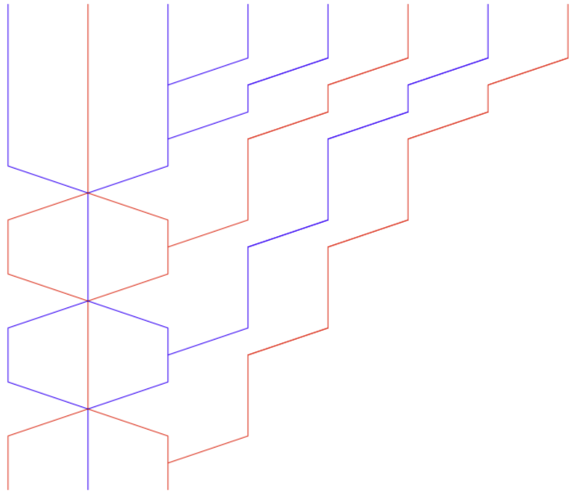


The solid crossings are underlined: $1\underline{1}2\underline{2}1\underline{1}\underline{2}$. ■

Solution: mutable, frozen, frozen, frozen ■

(iii) $\beta = \sigma_1 \sigma_2 \sigma_1 \sigma_1^2 \sigma_2 \sigma_1 \sigma_2 \in \mathrm{Br}_3^+$.

Solution:



The solid crossings are underlined: 121<u>11212</u>. ■

Solution: mutable, mutable, frozen, frozen, frozen ■

Exercise 7.2. In this exercise we will use the Cauchy-Binet formula, that says that if A, B are $n \times n$-matrices and $I, J \subseteq [n]$ are sets of the same size then

$$\Delta_{I,J}(AB) = \sum_{\substack{K \subseteq [n] \\ |K|=|I|=|J|}} \Delta_{I,K}(A)\Delta_{K,J}(B).$$

(a) As a warm-up: what does the Cauchy-Binet formula say when $|I| = |J| = 1$. What does it say when $|I| = |J| = n$?

Solution: When $|I| = |J| = 1$, this is the formula for matrix multiplication. When $|I| = |J| = n$, this says $\det(AB) = \det(A)\det(B)$. ■

(b) Let M be any $n \times n$ matrix . Let $j, i = 1, \ldots, n-1$ and let $I \subseteq [n]$ be a subset of size $|I| = i$. Show that:

$$\Delta_{I,[i]}(MB_j(z)) = \begin{cases} \Delta_{I,[i]}(M) & \text{if } j \neq i, \\ z\Delta_{I,[i]}(M) + \Delta_{I,s_i[i]}(M) & \text{if } j = i. \end{cases}$$

and

$$\Delta_{[i],I}(B_j(z)M) = \begin{cases} \Delta_{[i],I}(M) & \text{if } j \neq i, \\ z\Delta_{[i],I}(M) - \Delta_{s_i[i],I}(M) & \text{if } j = i, \end{cases}$$

where $s_i[i] = \{s_i(1), \ldots, s_i(i)\} = \{1, \ldots, i-1, i+1\}$.

Note: To see the general pattern, it may suffice to assume $i = 2$, $n = 4$.

Solution: We can check that

$$\Delta_{K,[i]}(B_j(z)) = \begin{cases} 1 & \text{if } K = [i], i \neq j \\ z & \text{if } K = [j], i = j \\ 1 & \text{if } K = s_j[j], i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus most of the terms in the Cauchy-Binet formula vanish. Taking the terms that do not vanish (depending on whether $i = j$) yields the desired equality. We can do something similar in the second case, just note that $\Delta_{[i],s_i[i]}(B_i(z)) = -1$. ■

(c) Use part (a) to show that, if $z, w \in \mathbb{C}$ then

$$\Delta_{[i],[i]}(MB_i(z))\Delta_{[i],[i]}(B_i(w)M) - \Delta_{[i],[i]}(M)\Delta_{[i],[i]}(B_i(w)MB_i(z)) = \det\begin{pmatrix} \Delta_{[i],[i]}(M) & \Delta_{s_i[i],[i]}(M) \\ \Delta_{[i],s_i[i]}(M) & \Delta_{s_i[i],s_i[i]}(M) \end{pmatrix}.$$

Solution: Expand out

$$\Delta_{[i],[i]}(MB_i(z)) = z\Delta_{[i],[i]}(M) + \Delta_{[i],s_i[i]}(M)$$

and

$$\Delta_{[i],[i]}(B_i(w)M) = w\Delta_{[i],[i]}(M) - \Delta_{s_i[i],[i]}(M)\,.$$

We also have

$$\Delta_{[i],[i]}(B_i(w)MB_i(z)) = w\Delta_{[i],[i]}(MB_i(z)) - \Delta_{s_i[i],[i]}(MB_i(z))$$

and

$$\Delta_{s_i[i],[i]}(MB_i(z)) = z\Delta_{s_i[i],[i]}(M) + \Delta_{s_i[i],s_i[i]}(M)\,.$$

Using these expansions, we can see that terms cancel on the LHS to yield the RHS. ■

Note that the right-hand side is independent of z, w! The *Desnanot-Jacobi* identity asserts that

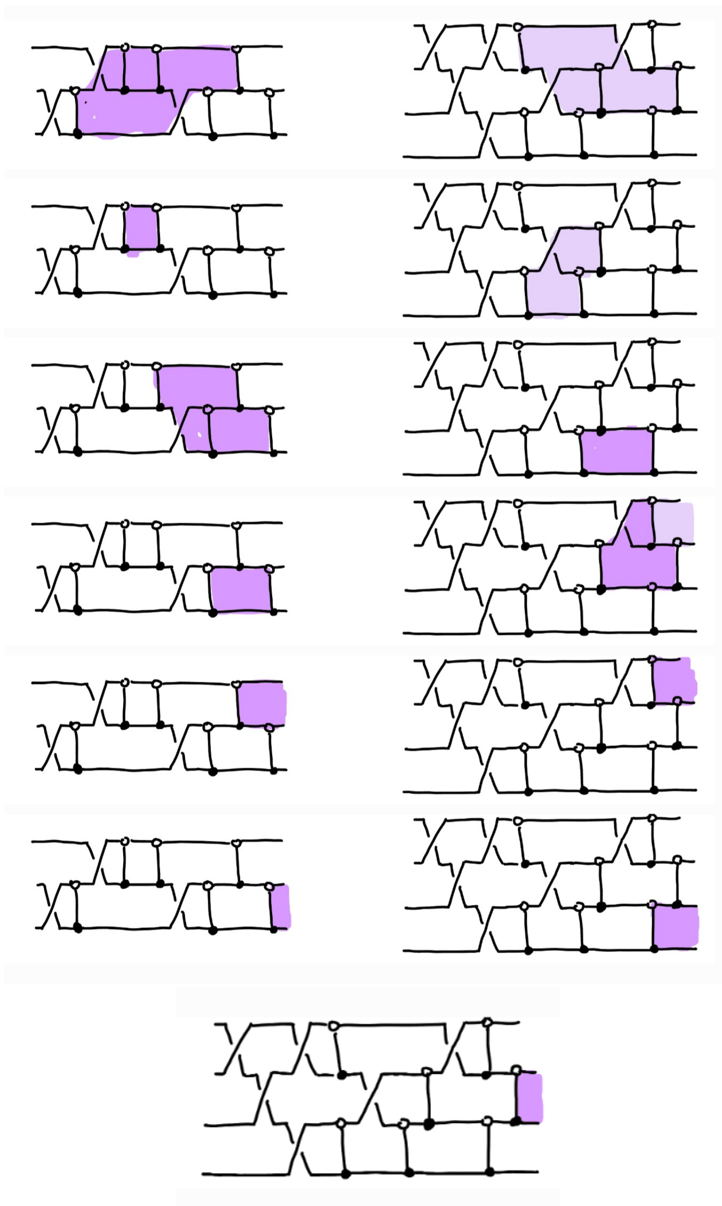
$$\det\begin{pmatrix} \Delta_{[i],[i]}(M) & \Delta_{s_i[i],[i]}(M) \\ \Delta_{[i],s_i[i]}(M) & \Delta_{s_i[i],s_i[i]}(M) \end{pmatrix} = \Delta_{[i-1],[i-1]}(M)\Delta_{[i+1],[i+1]}(M).$$

This identity will prove useful tomorrow.

Exercise 8.1. For each of the following braid words
(a) Draw its 3D plabic graph. (The answer is on the back of this page, but don't look yet!!)
(b) Draw the soap films for some solid crossings. You should use the graphs on the back of this page.

(i) $\beta = \sigma_1^2\sigma_2^3\sigma_1^2\sigma_2\sigma_1 \in \mathrm{Br}_3^+$.
(ii) $\beta = \sigma_3\sigma_2\sigma_1\sigma_3^2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3^2\sigma_1\sigma_2 \in \mathrm{Br}_4^+$.

Solution:



■

Based on these examples, what distinguishes the soap films S_d for mutable Deodhar hypersurfaces hypersurface from those for frozen Deodhar hypersurfaces?

Exercise 8.2. Let j be a hollow crossing, and let $\Delta^{\uparrow}$, $\Delta^{\downarrow}$, $\Delta^{\rightarrow}$, $\Delta^{\leftarrow}$ be the grid minors above, below, to the right, and to the left of crossing j in the 3D plabic graph. Use propagation rules to show that

$$\frac{\Delta^{\uparrow}\Delta^{\downarrow}}{\Delta^{\rightarrow}\Delta^{\leftarrow}} = 1.$$

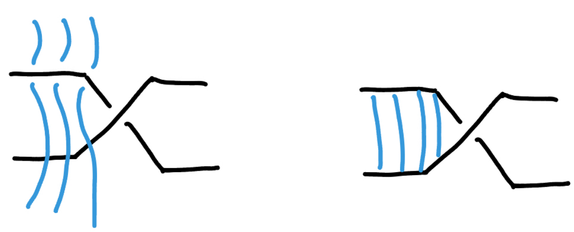
Solution: Look at all the ways that the soap films can propagate through a hollow crossing. There are 6 possibilities, using that some configurations are forbidden by Exercise 8.4. In each case, the soap film passes through the same number of left or right regions as it passes through the top or bottom regions. Thus the cluster variable x_k divides $\Delta^{\uparrow}\Delta^{\downarrow}$ with the same multiplicity that it divides $\Delta^{\rightarrow}\Delta^{\leftarrow}$. Since these are cluster monomials, we can conclude that $\Delta^{\uparrow}\Delta^{\downarrow} = \Delta^{\rightarrow}\Delta^{\leftarrow}$. ■

Lecture 8: additional exercises

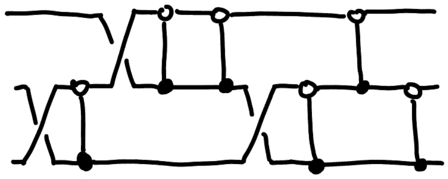
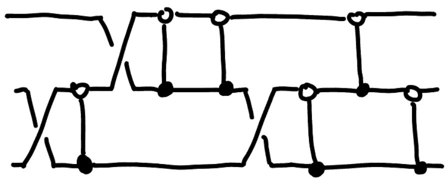
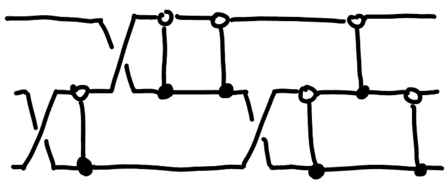
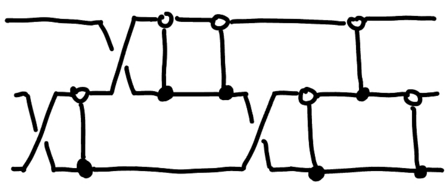
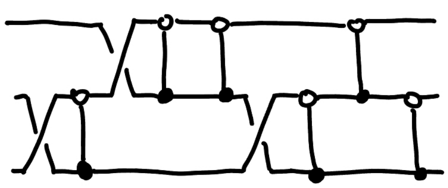
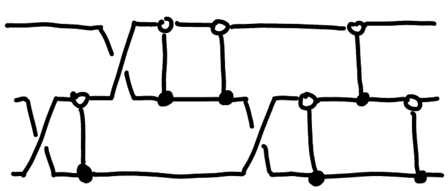
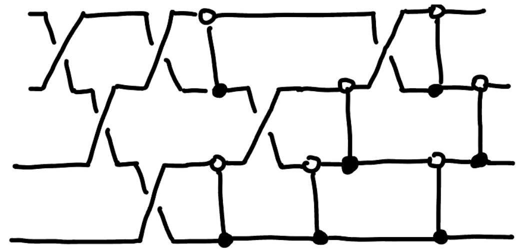
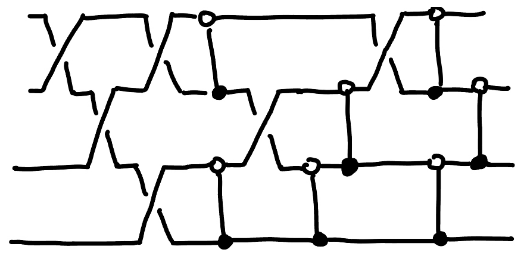
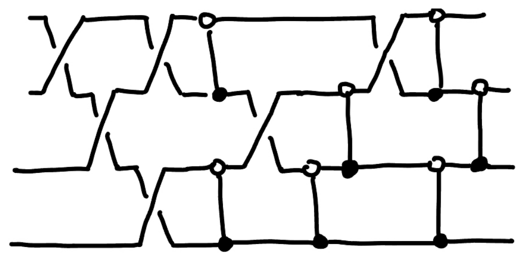
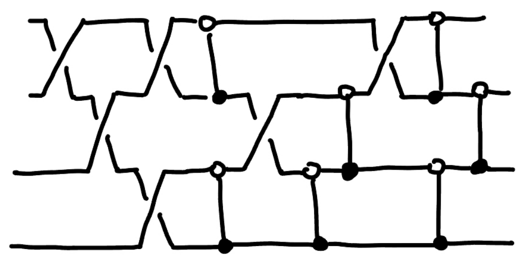
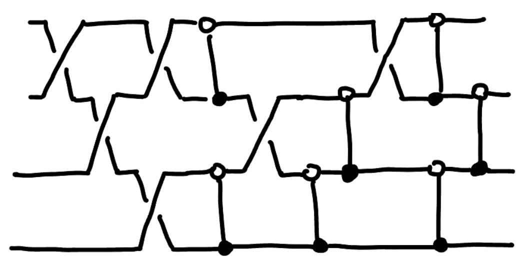
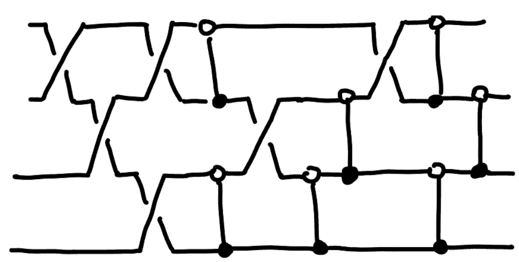
Exercise 8.3. Prove that the hollow crossings of a braid word β form the leftmost subexpression for w_0 in β. To be more precise, any other subexpression for w_0 will consist of a set of crossings that is lexicographically larger than the set of hollow crossings.

Solution: The hollow crossings detect when the length of the Demazure product increases as we take the left subwords of β. So any other choice of crossings would be lexicographically larger, since we'd be choosing something to the right of the corresponding hollow crossing. ■

Exercise 8.4. The propagation rules implicitly assert that soap films can always "pass through" hollow crossings undisturbed. That is, you never see either of the local configurations below.



Prove that these configurations never occur using the skew-shape lemma.

For a braid word $\mathbf{i}$, we consider the double Bott-Samelson variety $\mathrm{BS}(\mathbf{i})$ with the initial seed $(x_{\mathbf{i}}, Q_{\mathbf{i}})$ described in lecture.

Exercise 9.1. Take the braid word on six strands 142145. Note that the letter 3 is missing from this braid word. Show that

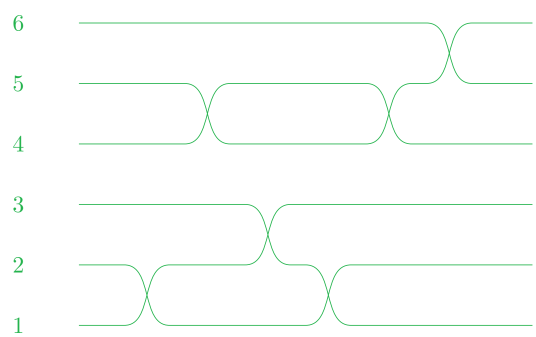
$$\mathrm{BS}(142145) \cong \mathrm{BS}(121) \times \mathrm{BS}(445).$$

You could easily do this using cluster algebras, but try to do it without. The upshot is that in many arguments we can assume that every letter in $\{1, 2, \ldots, n-1\}$ actually appears at least once in a braid word.

Solution: Note that

$$BS(142145) := \{(\mathcal{F}^0 = \mathcal{F}^{std}, \mathcal{F}^1, \ldots, \mathcal{F}^6) : \mathcal{F}^{std} \xrightarrow{\sigma_1} \mathcal{F}^1 \xrightarrow{\sigma_4} \mathcal{F}^2 \xrightarrow{\sigma_2} \mathcal{F}^3 \xrightarrow{\sigma_1} \mathcal{F}^4 \xrightarrow{\sigma_4} \mathcal{F}^5 \xrightarrow{\sigma_5} \mathcal{F}^6 \xrightarrow{w_0} \mathcal{F}^{anti}\}$$

Visually, the braid looks like



Since, the crossing σ_3 is not included in the braid we can "separate" the braid into two disjoint braids. Now, we can decompose the sequences of flags as

$$\{\mathcal{F}^{std} \xrightarrow{\sigma_1} \mathcal{F}^1 \xrightarrow{\sigma_2} \widetilde{\mathcal{F}}^2 \xrightarrow{\sigma_1} \widetilde{\mathcal{F}}^3 \xrightarrow{w_0} \mathcal{F}^{anti}\} = BS(121)$$

$$\{\mathcal{F}^{std} \xrightarrow{\sigma_4} \widetilde{\mathcal{F}}^1 \xrightarrow{\sigma_4} \widetilde{\widetilde{\mathcal{F}}}^2 \xrightarrow{\sigma_5} \widetilde{\widetilde{\mathcal{F}}}^3 \xrightarrow{w_0} \mathcal{F}^{anti}\} = BS(445)$$

PLEASE CHECK ■

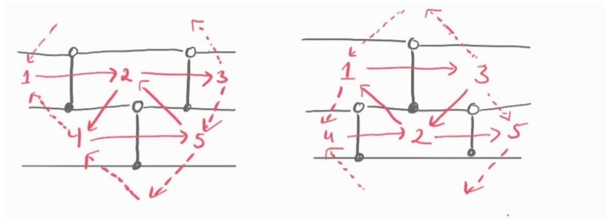
Exercise 9.2. Assume $\mathbf{i} = \cdots i\underline{i}i \cdots$. Show explicitly that the variable obtained after mutating at the vertex corresponding to the underlined $\underline{i}$ is regular.

Exercise 9.3. Suppose $|i - j| = 1$. Suppose also that $\mathbf{i}$ and $\mathbf{i}'$ are braid letters of the following form

$$\mathbf{i} = \mathbf{j_1}\underline{i}ji\mathbf{j}_2, \qquad \mathbf{i}' = \mathbf{j}_1\underline{j}ij\mathbf{j_2}.$$

Show that $Q_{\mathbf{i}}$ and $Q_{\mathbf{i}'}$ differ by a mutation, precisely at the vertex corresponding to the underlined letters above.

Solution:



Observe that these quivers differ by a mutation at vertex 2.

■

Exercise 9.4. Consider the braid word $\mathbf{i} = 1^n$, that is, $\mathbf{i} = \underbrace{11\cdots1}_{n \text{ times}}$. Use the Starfish lemma and Exercise 9.2 above to show that $(x_{\mathbf{i}}, Q_{\mathbf{i}})$ indeed endows $\mathbb{C}[\mathrm{BS}(\mathbf{i})]$ with a cluster structure.

Exercise 9.5. Let $\tilde{T} \subseteq \mathrm{GL}(k)$ be the torus of diagonal matrices, and let T be the quotient of $\tilde{T}$ by the subgroup of scalar matrices. Note that the torus $\tilde{T}$ acts on the flag variety, and this action factors through its quotient T.

(a) Show that T preserves relative position, i.e., if $\mathcal{F}^1_\bullet \xrightarrow{w} \mathcal{F}^2_\bullet$ and $t \in T$, then $t.\mathcal{F}^1_\bullet \xrightarrow{w} t.\mathcal{F}^2_\bullet$. Conclude that T acts naturally on $\mathrm{BS}(\beta)$, for any k-stranded braid β.

(b) Let β be a braid such that its projection to S_k is a single k-cycle. Show that T acts freely on $\mathrm{BS}(\beta)$. For this, it may be useful to explicitly give the action of T on coordinates $(z_1, \ldots, z_k)$.

In fact, the converse of (b) is also true: T acts freely on $\mathrm{BS}(\beta)$ if and only if β projects to a k-cycle.

Note that by the same reasoning, one gets an action of T on the braid variety $X(\beta)$ for any braid β. But the converse of (b) is not valid for general braid varieties – it is one of the many results that are valid for double Bott-Samelson varieties, but not general braid varieties.

Lecture 10: main exercises

Exercise 10.1. Recall that $\mathbb{C}[\mathrm{BS}(\mathbf{i})] \cong \mathbb{C}[z_1, \dots, z_r][f_1^{-1}, \dots, f_{k-1}^{-1}]$, where $f_1, \dots, f_k$ are the principal minors of the matrix $B_{\mathbf{i}}(z_1, \dots, z_r)$. Consider the cluster structure on $\mathbb{C}[\mathrm{BS}(\mathbf{i})]$ constructed in the lecture. Show that all cluster variables (not just those in the initial seed) in $\mathbb{C}[\mathrm{BS}(\mathbf{i})]$ belong to $\mathbb{C}[z_1, \dots, z_r]$, that is, they do not involve negative powers of frozen variables.

Exercise 10.2. Let $k < n$, and consider the variety of n-tuples of points $(v_1, \ldots, v_n) \in (\mathbb{C}^k)^n$ such that $v_i, v_{i+1}, \ldots, v_{i+k-1}$ are linearly independent for every $i = 1, \ldots, n$, where the indices are taken mod n. Let $\Pi^\circ_{k,n}$ be the subvariety consisting of those tuples $(v_1, \ldots, v_n)$ such that $v_{n-k+1} = e_1, v_{n-k+2} = e_2, \ldots, v_n = e_k$. (If you want, you can assume $k = 4, n = 8$ – once you have cracked this case the general pattern should be clear)

(a) Consider the k-stranded braid $\beta_{k,n} = (\sigma_{k-1} \cdots \sigma_1)^{n-k}$. Exhibit a map $\Pi^\circ_{k,n} \to \mathrm{BS}(\beta_{k,n})$.

(b) Show that $\Pi^\circ_{k,n}$ is isomorphic to the open positroid variety Π^e_w, where $e \in S_n$ is the identity and $w = (n-k+1) \cdots n12 \cdots (n-k)$.

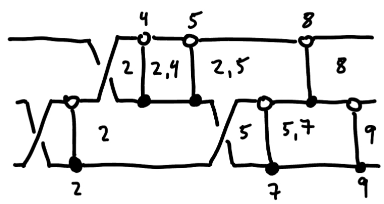
Note that, by dimension reasons, the map you constructed in a) cannot be an isomorphism – the positroid variety has dimension $k(n-k)$ while the double Bott-Samelson variety $\mathrm{BS}(\beta_{k,n})$ has dimension $(k-1)(n-k)$. In fact, $\Pi^\circ_{k,n} \cong \mathrm{BS}(\beta_{k,n}) \times (\mathbb{C}^\times)^{n-k}$, a fact that can be explained using cluster algebras as follows.

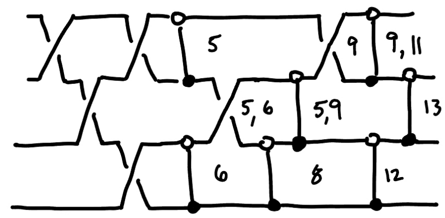
(c) Go to Page 25 in `https://arxiv.org/pdf/2008.09189` and see the quiver and cluster variables there for the case $k = 3, n = 7$ (for example, the variable (457) should be interpreted as being the determinant of $\begin{bmatrix} v_4 \\ v_5 \\ v_7 \end{bmatrix}$). Setting the upper left corner equal to 1, we obtain a cluster structure on $\Pi^\circ_{3,7}$. Compare this quiver with the one you obtain from $\mathrm{BS}(\beta_{3,7})$. How are they different? Now (the proof of) Proposition 5.11 in `https://arxiv.org/pdf/1604.06843` implies the desired result (granted, this proposition is not easy to read!)

Exercise 11.1. For each of the following braid words, draw the quiver Q_β.

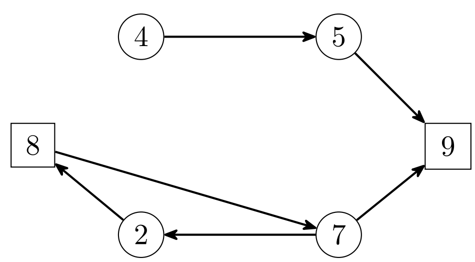
(i) $\beta = \sigma_1^2\sigma_2^3\sigma_1^2\sigma_2\sigma_1 \in \mathrm{Br}_3^+$.
(ii) $\beta = \sigma_3\sigma_2\sigma_1\sigma_3^2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_3^2\sigma_1\sigma_2 \in \mathrm{Br}_4^+$.

To make this a little faster, here are the 3D plabic graphs, with each region labeled by the cluster variables that appear there.

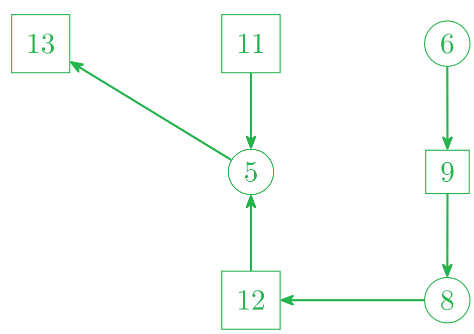




Solution: ■



Solution:

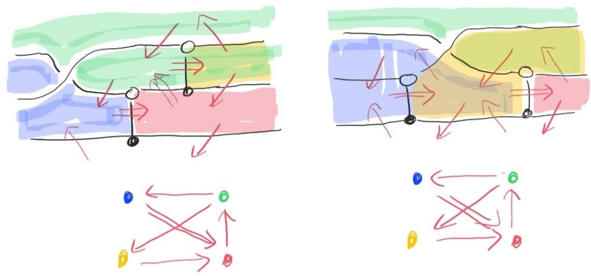


■

Exercise 11.2. Suppose β, β' are related by a braid move, that is $\beta = \ldots iji \ldots$ and $\beta' = \ldots jij \ldots$ where $|i-j| = 1$. Convince yourself of the following.

(a) If any letters involved in the braid move are hollow, then Q_β and $Q_{\beta'}$ are related by relabeling.

Solution:



Here is one case of many. We can draw the soap bubbles for various regions and look at the local contribution to the quiver. By relabeling as follows, we can see that the resulting quiver is the same. ■

(b) If all letters involved in the braid move are solid, then $Q_{\beta'}$ is a relabeling of $\mu_r(Q_\beta)$, where the leftmost letter in the braid move is letter r of β.

Solution: See the solution to Exercise 9.3. Note that we performed a relabeling of the regions to identify the two quivers on the plabic fence. Draw some soap films on this and check that the corresponding contributions to the quivers are the same. ■

Lecture 11: additional exercises

Exercise 11.3. Show that the half-arrow and surface description of the quiver agree. Hint: For the surface description, if you force the curves C_d to lie on the graph G_β, then $C_d \cap C_{d'}$ is a disjoint union of paths. The signed intersection number of the curves can be computed one path at a time.

Exercise 12.1. One tool in the proof that $\mathbb{C}[X(\beta)]$ is a cluster algebra is *rotation*. Suppose $\beta = \gamma i$ for γ a positive braid word on $[n]$. Define $i^* := n - i$ and let $\beta' = i^*\gamma$. Construct an isomorphism $X(\beta) \to X(\beta')$.

Solution: Perform rainbow closure of the braid and bring σ_i to beginning of word to obtain $\beta' = i^*\gamma$ where $i^* = w_o[i]$.
Can describe in flags. ■

Exercise 12.2. Suppose $\beta = \gamma ii$ and say β has r letters. Suppose $r - 1 \notin J_\beta$, that is, the second-to-last i is hollow.

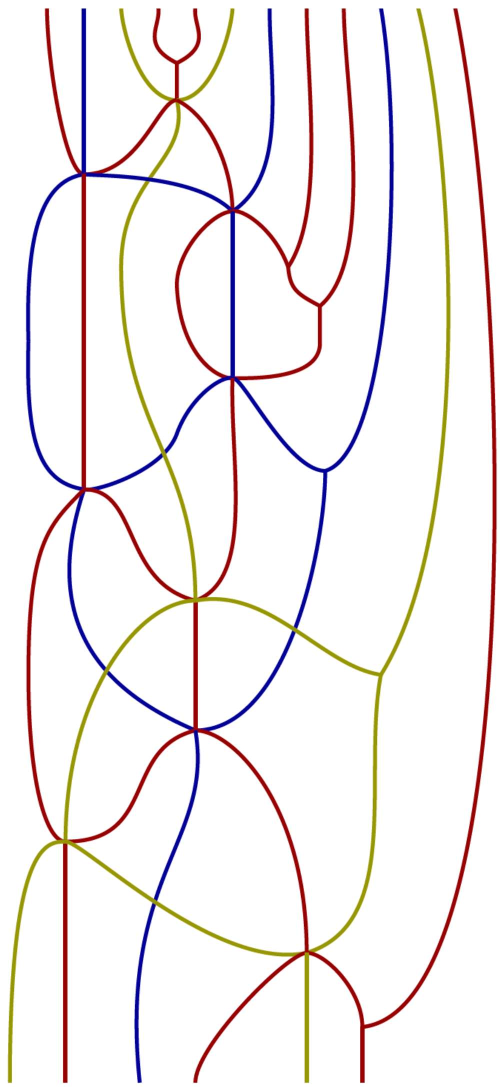
(a) Express x_r, the cluster variable associated to the final letter of β in terms of grid minors.
Solution: In this case, $\Delta_r = x_r \cdot \prod_j x_j$, where the product is over all x_j that pass through the chamber for Δ_r. These must also be frozen, so we can divide Δ_r by these. ■

(b) Construct an isomorphism $X(\beta) \to X(\gamma i) \times \mathbb{C}^\times$ so that the coordinate of $\mathbb{C}^\times$ is x_r.

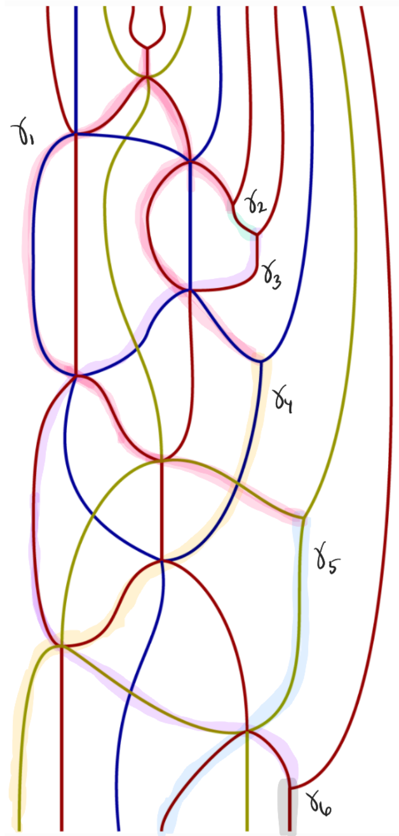
(c) Verify that if you remove vertex r from Q_β and variable x_r from the cluster $\mathbf{x}_\beta$, you obtain the seed $\Sigma_{\gamma i}$.

Solution: When we construct the quiver for Q_β, we can see that x_r is an isolated frozen vertex. So deleting it from the quiver/cluster algebra are equivalent. ■

Exercise 13.1. In the weave below, identify all Lusztig cycles and compute their pairwise intersections.



Solution:



$$\gamma_1 \circ \gamma_2 = 1$$

$$\gamma_1 \circ \gamma_3 = \frac{1}{2}\left(\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix} - \begin{vmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 0\end{vmatrix}\right) + \frac{1}{2}\left(\begin{vmatrix}1 & 1 & 1\\ 1 & 0 & 0\\ 0 & 0 & 1\end{vmatrix} - \begin{vmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 1 & 0 & 0\end{vmatrix}\right)$$

$$= \frac{1}{2}(-1-1) + \frac{1}{2}(-1-1) = -2$$

$$\gamma_1 \circ \gamma_4 = 1$$

$$\gamma_1 \circ \gamma_5 = 1$$

$$\gamma_1 \circ \gamma_6 = 0$$

$$\gamma_2 \circ \gamma_3 = 1$$

$$\gamma_2 \circ \gamma_4 = 0$$

$$\gamma_2 \circ \gamma_5 = 0$$

$$\gamma_2 \circ \gamma_6 = 0$$

$$\gamma_3 \circ \gamma_4 = 1$$

$$\gamma_3 \circ \gamma_5 = -1$$

$$\gamma_3 \circ \gamma_6 = -1$$

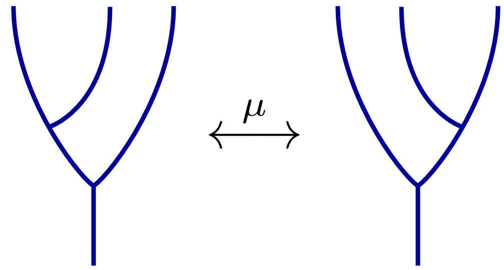
$$\gamma_4 \circ \gamma_5 = 0$$

$$\gamma_4 \circ \gamma_6 = 0$$

$$\gamma_5 \circ \gamma_6 = 0$$

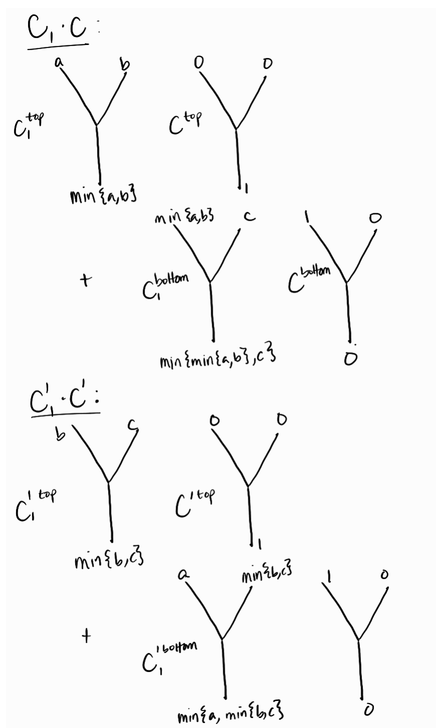
CHECK SIGNS ■

Exercise 13.2. Suppose that two weaves $\mathfrak{W}, \mathfrak{W}'$ agree everywhere except for a weave from σ^3 to σ:



Let C_1 (resp. C_1') be a Lusztig cycle which starts above this fragment and has values a, b, c on top edges. Let C (resp. C') be a short cycle supported on the internal edge. Prove $C_1 \cdot C = -C_1' \cdot C'$ by computing both sides.

Solution:



$$
\begin{aligned}
C_1 \cdot C &= \begin{vmatrix} 1 & 1 & 1 \\ a & \min\{a,b\} & b \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ \min\{a,b\} & \min\{\min\{a,b\},c\} & c \\ 1 & 0 & 0 \end{vmatrix} \\
&= (-b+a) + (c - \min\{\min\{a,b\},c\}) \\
&= a - b + c - \min\{\min\{a,b\},c\} \\
-C_1' \cdot C' &= -\left(\begin{vmatrix} 1 & 1 & 1 \\ b & \min\{b,c\} & c \\ 0 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a & \min\{a,\min\{b,c\}\} & \min\{b,c\} \\ 0 & 0 & 1 \end{vmatrix} \right) \\
&= -\left((b-c) + (\min\{a,\min\{b,c\}\} - a)\right) \\
&= a - b + c - \min\{a, \min\{b,c\}\}
\end{aligned}
$$

■

Exercise 13.3. A **semifield** is a set with addition $\oplus$, multiplication $\otimes$ and division (but no subtraction) satisfying the usual axioms.

(a) Prove that $\mathbb{Z} \cup \{\infty\}$ with operations $a \oplus b = \min(a, b)$ and $a \otimes b = a + b$ is a semifield (sometimes called tropical semifield). Write a formula for division in this semifield.

(b) Prove that the set R of rational functions $p(t)/q(t)$ where $p(t), q(t)$ are polynomials with nonnegative integer coefficients, and usual operations, is a semifield.

(c) Prove that there is a unique semifield homomorphism $\nu : R \to \mathbb{Z} \cup \{\infty\}$ such that $\nu(t^a) = a$.

Exercise 14.1. Suppose

$$B_i(z_1)\chi_i(u_1)B_{i+1}(z_2)\chi_{i+1}(u_2)B_i(z_3)\chi_i(u_3) = B_{i+1}(z_1')\chi_{i+1}(u_1')B_i(z_2')\chi_i(u_2')B_{i+1}(z_3')\chi_i(u_3')$$

where all u, u' are invertible. Prove that z_1', z_2', z_3' are uniquely determined by z_1, z_2, z_3 and $u_1, u_2, u_3, u_1', u_2', u_3'$ provided that

$$u_1u_2 = u_2'u_3' \text{ and } u_2u_3 = u_1'u_2'.$$

Solution: The product on the left side is

$$\begin{bmatrix} u_3(u_1z_1z_3 - u_1^{-1}u_2z_2) & -u_1z_1u_3^{-1} & (u_1u_2)^{-1} \\ u_1u_3z_3 & -u_1u_3^{-1} & 0 \\ u_2u_3 & 0 & 0 \end{bmatrix}$$

The product on the right hand side is

$$\begin{bmatrix} u_2'z_2' & -u_3'z_3'(u_2')^{-1} & (u_2'u_3')^{-1} \\ u_1'u_2'z_1' & -u_3'(u_1')^{-1} & 0 \\ u_1'u_2' & 0 & 0 \end{bmatrix}$$

From this, we see

$$\begin{aligned} u_1z_1u_3^{-1} &= u_3'z_3'(u_2')^{-1} \\ u_1u_3z_3 &= u_1'u_2'z_1' \\ u_3(u_1z_1z_3 - u_1^{-1}u_2z_2) &= u_2'z_2' \end{aligned}$$

From this, we get

$$\begin{aligned} z_1' &= \frac{u_1z_3}{u_2} \\ z_2' &= \frac{u_3(u_1^2z_1z_3 - u_2z_2)}{u_1u_2'} \\ z_3' &= \frac{u_1z_1u_2'}{u_3u_3'} \end{aligned}$$

■

Exercise 14.2. Assume that $\widetilde{z}_2 \neq 0$. Prove that

$$B_i(z_1)\chi_i(u_1)B_i(z_2)\chi_i(u_2) = B_i(z)\chi_i(u)U$$

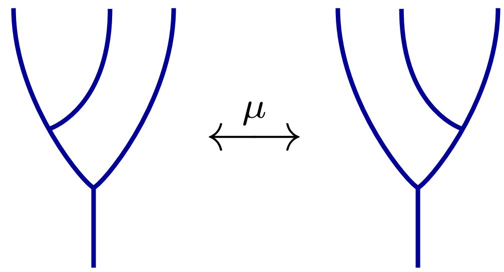
where $z = z_1 - u_1^{-2}\widetilde{z}_2^{\,-1}$, $u = z_2u_1u_2$ and U is an upper unitriangular matrix.

Solution: The matrix product on the left is

$$\begin{bmatrix} u_2(u_1z_1z_2 - u_1^{-1}) & -u_1z_1u_2^{-1} \\ u_1u_2z_2 & -u_1u_2^{-1} \end{bmatrix}$$

Setting $U = \begin{bmatrix} 1 & -u_2^{-2}z_2^{-1} \\ 0 & 1 \end{bmatrix}$ and $\tilde{z}_2 = z_2$, the matrix product on the right is the same. ■

Exercise 14.3. Compute the two pairs of cluster variables for two Demazure weaves from σ^3 to σ using problem 2. Verify that these are regular functions and that two seeds are related by mutation. *Hint: you will need to push an upper-triangular matrix to the right for one of the weaves.*



Solution: This computation is at the end of the Lecture 14 notes. In particular, we get $A_1 = z_3$, $A_2 = z_2 z_3 - 1$, $A_1' = z_2$, and $A_2' = z_2 z_3 - 1$. So we can see $A_1' = \frac{1+A_2}{A_1}$ ■

Exercise 15.1. For an integer x denote $[x]_+ = \max(x,0)$ and $[x]_- = \min(x,0)$. Prove that

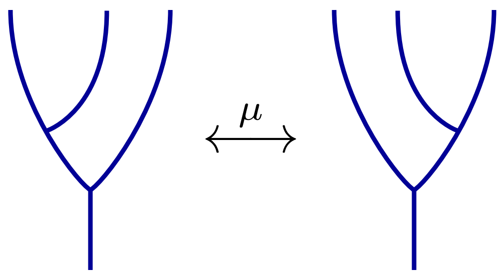
$$[b-a-c+\min(a,b,c)]_- = -[a+c-b-\min(a,b,c)]_+ = \min(a,b)-c-a+\min(b,c),$$

$$[b-a-c+\min(a,b,c)]_+ = -[a+c-b-\min(a,b,c)]_- = b+\min(a,b,c)-\min(a,b)-\min(b,c)$$

Hint: use tropicalization!

Solution: This is in https://arxiv.org/pdf/2207.11607, Lemma 4.11. ■

Exercise 15.2. Consider the following figure as part of the bigger weave. Suppose that the Lusztig cycle C_i (not corresponding to trivalent vertices in the picture) has weights a_i, b_i, c_i at the top, and A_i the corresponding cluster variable. Let v_1 (resp. v_2) denote the top (resp. bottom trivalent vertex), A_{v_1} (resp. A_{v_2}) the corresponding cluster variables in the left weave, and $\overline{A_{v_1}}$ (resp. $\overline{A_{v_2}}$) the corresponding cluster variables in the right weave.



(a) In the left weave, prove that

$$A_{v_1} = z_2 \prod A_i^{a_i+b_i-\min(a_i,b_i)}$$

and

$$A_{v_2} = \left(z_2 z_3 - \prod A_i^{-2b_i}\right) \prod A_i^{a_i+b_i+c_i-\min(a_i,b_i,c_i)}.$$

(b) In the right weave, prove that:

$$\overline{A_{v_1}} = z_3 \prod A_i^{b_i+c_i-\min(b_i,c_i)}, \quad \text{and} \quad \overline{A_{v_2}} = A_{v_2}.$$

(c) Prove that

$$A_{v_1}\overline{A_{v_1}} = A_{v_2} \prod A_i^{b_i+\min(a_i,b_i,c_i)-\min(a_i,b_i)-\min(b_i,c_i)} + \prod A_i^{a_i+c_i-\min(a_i,b_i)-\min(b_i,c_i)}$$

Use Exercise 15.1 to conclude that A_{v_1} and $\overline{A_{v_1}}$ are related by mutation.

Solution: This is in https://arxiv.org/pdf/2207.11607, Lemma 5.16. ■

Lecture 16: main exercises

Exercise 16.1. Consider the quiver $\boxed{a} \to ① \to \boxed{b}$.

(a) Show that its cluster algebra A is isomorphic to $\mathbb{C}[x_1, x_1', y_a^{\pm 1}, y_b^{\pm 1}]/(x_1x_1' - y_a - y_b)$. (cf. Exercise 5.1).

Solution: Here, we obtain two seeds with the mutation rule that $x_1x_1' = y_a + y_b$. Therefore, the cluster algebra $\mathcal{A} \cong \mathbb{C}[x_1, x_1', y_a^{\pm}, y_b^{\pm}]/(x_1x_1' = y_a + y_b) \cong \mathbb{C}[x_1, x_1', y_a^{\pm}, y_b^{\pm}]/(x_1x_1' - y_a - y_b)$. ■

(b) Let $z := y_1^{-1}x_1 \in A$. Show that the following seed defines a cluster structure on A:

$$\boxed{y_a} \qquad ⓩ \to \boxed{y_a^{-1}y_b}$$

Thus, the same algebra A can have many different cluster structures.

Solution: The corresponding cluster algebra is $\mathcal{A} \cong \mathbb{C}[z, z', y_a^{\pm}, (y_a^{-1}y_b)^{\pm}]/(zz' = 1 + y_a^{-1}y_b)$, then use to solve for variables. ■

Exercise 16.2. Recall that we set

$$x_j(z) = \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & 1 & z & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}, \qquad \dot{s}_j = \begin{pmatrix} 1 & \cdots & & & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \cdots & & & \cdots & 1 \end{pmatrix}$$

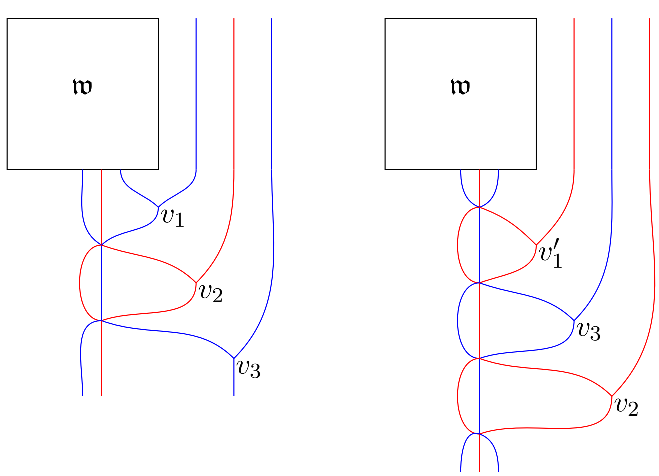
so that $B_j(z) = x_j(z)\dot{s}_j$. Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word. Show that we can write

$$B_{i_1}(z_1)\chi_{i_1}(u_1) \cdots B_{i_\ell}(z_\ell)\chi_{i_\ell}(u_\ell) = x_{i_1}(z_1)\dot{s}_{i_1}\chi_{i_1}(u_1) \cdots x_{i_\ell}(z_\ell)\dot{s}_{i_\ell}\chi_{i_\ell}(u_\ell) = g\left(\dot{s}_{i_1}\chi_{i_1}(u_1) \cdots \dot{s}_{i_\ell}\chi_{i_\ell}(u_\ell)\right)$$

where g is an upper uni-triangular matrix.

Hint: Show that if $s_{i_1} \cdots s_{i_\ell}$ is reduced, then $\left(\dot{s}_{i_1} \cdots \dot{s}_{i_{\ell-1}}\right) x_{i_\ell}(z) \left(\dot{s}_{i_1} \cdots \dot{s}_{i_{\ell-1}}\right)^{-1}$ is upper uni-triangular.

Exercise 16.3. Consider the following weaves



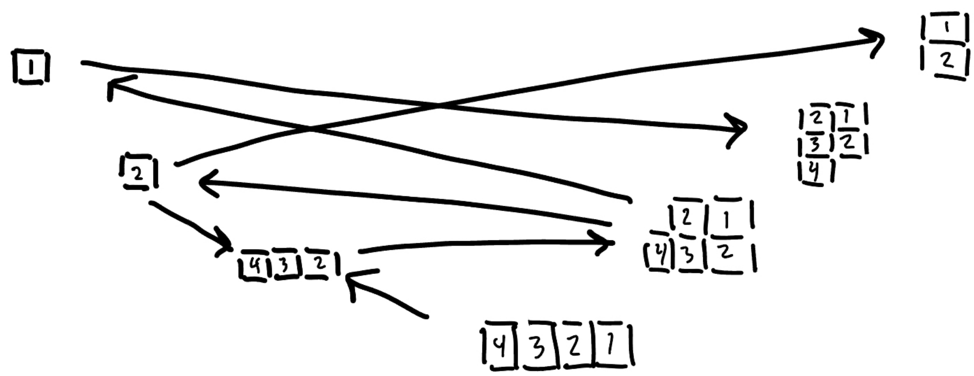
Show that the corresponding quivers are mutation equivalent, with mutation at the vertex v_1. Compare to Exercise 11.2 from Thursday's first lecture. Note that this is also similar to Exercise 9.2 in Wednesday's first lecture, but here we are not assuming we are in the double Bott-Samelson case.

Lecture 17: main exercises

Exercise 17.1. Verify that the skew-shape modules are preprojective algebra modules. That is, verify that the maps ψ_i, ψ_i^* satisfy

$$\sum_{i=1}^{n-1} \psi_i \psi_{i^*} - \psi_{i^*} \psi_i = 0.$$

Exercise 17.2. Choose two arrows in the endomorphism quiver below. For each of your chosen arrows, explicitly find a map between the two modules and check that it is a morphism of preprojective algebra modules.



Lecture 17: additional exercises

Exercise 17.3. Suppose $\nu \subset \mu \subset \lambda$, so the skew shape λ/ν contains λ/μ and μ/ν. Show that there is an injective morphism $M_{\lambda/\mu} \hookrightarrow M_{\lambda/\nu}$ and a surjective morphism $M_{\lambda/\nu} \twoheadrightarrow M_{\mu/\nu}$. (These maps are enough to give the morphisms in the example from lecture, but in general you need more.)

Exercise 18.1. For each n, find a braid β on n strands such that the cluster structure on $X(\beta)$ has $\binom{n}{2}$ frozen variables. Hint: see the next exercise

Solution: Consider the reduced word $w_0 = (1, 2, \cdots, n-1, 1, 2, \ldots, n-2, \cdots, 1, 2, 1)$. Then let β be the word obtained by doubling every letter in w_0 (as in the next exercise). The inductive weave will consist only of 3-valent vertices between each pair of identical letters, with the bottom of each 3-valent vertex reaching the bottom of the diagram. Since there are $\binom{n}{2}$ trivalents, this yields $\binom{n}{2}$ frozen variables. ■

Exercise 18.2. Let $\beta = 112211$. Recall the torus T that is the quotient of the torus of diagonal matrices in GL(3) by the torus of scalar matrices. Show that T acts freely on $X(\beta)$, but neither β nor βw_0 project to a 3-cycle. For a description on when T acts freely on $X(\beta)$, see 6.3 in `https://arxiv.org/pdf/2402.16970`.