EXERCISES FOR LECTURE 1

1. Main exercise

Exercise 1. (1) If G is an algebraic group, a *character* is a morphism of algebraic groups $G \to \mathbb{G}_{\mathrm{m}}$. Given a character χ, a *weight vector* of weight χ in a representation V is a vector v such that $g \cdot v = \chi(g)v$ for all $g \in G$. If $v \in V$ is a weight vector, compute the image of v under the coaction morphism

$$V \to V \otimes \mathcal{O}(G).$$

(2) Let $m \geq 1$, and consider the algebraic group

$$G = (\mathbb{G}_{\mathrm{m}})^m = \mathbb{k}^\times \times \cdots \times \mathbb{k}^\times.$$

The corresponding Hopf algebra is $\mathbb{k}[x_i^{\pm 1} : 1 \leq i \leq m]$, with the comultiplication determined by

$$\Delta(x_i) = x_i \otimes x_i.$$

Its group $X^*(G)$ of characters (i.e. morphisms of algebraic groups from G to $\mathbb{G}_{\mathrm{m}}$) identifies with $\mathbb{Z}^m$, where $(\lambda_1, \dots, \lambda_m)$ corresponds to

$$(t_1, \dots, t_m) \mapsto \prod_i (t_i)^{\lambda_i}.$$

(a) Show that every representation of this group is the sum of its *weight spaces*, i.e. if $V \in \mathrm{Rep}(G)$, then

$$V = \bigoplus_{\lambda \in X^*(G)} V_\lambda \quad \text{where } V_\lambda = \{v \in V \mid \in G,\ g \cdot v = \lambda(g)v\}.$$

The *weights* of V are the elements $\lambda \in X^*(G)$ such that $V_\lambda \neq 0$.

(b) Given representations V, V' of G, determine the decomposition in weight spaces of $V \otimes V'$ in terms of those of V and V'.

(3) Consider the case $G = \mathrm{SL}_2(\mathbb{k})$, with the choices of B and T as in the lecture, and assume that $\mathrm{char}(\mathbb{k}) = 0$.

(a) Recall, for any $m \geq 0$, the representation

$$V_m = \mathbb{k}_m[x, y]$$

considered in the lecture. Compute the weights of T on V_m.

(b) Show that each V_m is a simple representation.

(c) Determine, for m in $\mathbb{Z}_{\geq 0}$, the decomposition of $V_1 \otimes V_m$ as a direct sum of simple representations. (*Hint*: use semisimplicity and the description of the weights of these representations.)

2. Additional exercises

Exercise 2. (1) In this question we assume that $G = \mathrm{SL}_2(\Bbbk)$. Determine, for m_1, m_2 in $\mathbb{Z}_{\geq 0}$, the decomposition of $V_{m_1} \otimes V_{m_2}$ as a direct sum of simple representations.

(2) Now we take $G = \mathrm{SL}_n(\Bbbk)$, with the choices of B and T as in the lecture. The Lie algebra $\mathfrak{g}$ identifies with $\{M \in \mathrm{M}_n(\Bbbk) \mid \mathrm{tr}(M) = 0\}$, and the adjoint action is given by conjugation of matrices.

(a) For $i \in \{1, \ldots, n-1\}$, we set

$$\omega_i = [\varepsilon_1 + \cdots + \varepsilon_i] \quad \in \mathbb{Z}^n/\Delta\mathbb{Z}.$$

Show that the map

$$(\lambda_1, \ldots, \lambda_{n-1}) \mapsto \sum_i \lambda_i \cdot \omega_i$$

defines a bijection $(\mathbb{Z}_{\geq 0})^{n-1} \xrightarrow{\sim} \mathbf{X}^+$.

(b) We denote by $V = \Bbbk^n$ the natural representation. For $i \in \{1, \cdots, n-1\}$, determine the weights of T on $\wedge^i V$.

(c) Show that each $\wedge^i V$ is a simple representation. (Under the bijection in Chevalley's theorem, this representation corresponds to ω_i.)

Exercise 3. (1) Show that when $G = \mathrm{GL}_n$ with the choice of T as in the lecture, the canonical exact sequence

$$1 \to T \to N_G(T) \to W \to 1$$

admits a canonical splitting.

(2) Show that when $G = \mathrm{SL}_2$ with T as in the lecture, the exact sequence of the preceding question does *not* admit a splitting.

Exercise 4. Let G be an algebraic group over $\Bbbk$. Denote by $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ the comultiplication morphism, by $S : \mathcal{O}(G) \to \mathcal{O}(G)$ the map sending a function f to the function $g \mapsto f(g^{-1})$, and by $\varepsilon : \mathcal{O}(G) \to \Bbbk$ the map sending f to $f(1)$. Show that these maps satisfy the following conditions:

- Δ, S and ε are algebra morphisms;
- $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \circ \mathrm{id}) \circ \Delta$ (co-associativity axiom);
- $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \circ \mathrm{id}) \circ \Delta$ (co-unit axiom);
- $m \circ (S \otimes \mathrm{id}) \otimes \Delta = \varepsilon(-) \cdot 1 = m \circ (\mathrm{id} \otimes S) \otimes \Delta$ (antipode axiom).

(These are the axioms in the definition of a *Hopf algebra*.)