EXERCISES FOR LECTURE 10

1. Main exercise

Exercise 1. In this exercise we assume that $G = \mathrm{PGL}_2(\mathbb{C})$, with T the maximal torus consisting of (images of) diagonal matrices and B the Borel subgroup consisting of (images of) upper triangular matrices. In this case the lattice $\mathbf{X}^\vee$ identifies with $\mathbb{Z}$ via the map

$$n \mapsto \left(z \mapsto \begin{bmatrix} z^n & 0 \\ 0 & 1 \end{bmatrix} \right),$$

and $\mathbf{X}^\vee_+$ corresponds to $\mathbb{Z}_{\geq 0}$. Under this identification, the pairing with 2ρ is given by $n \mapsto n$; as a consequence $\dim(\mathrm{Gr}^n) = n$.

(1) Prove that for any $n \geq 0$ we have:
   (a) $\mathrm{IC}_n = \underline{\mathbb{C}}_{\overline{\mathrm{Gr}^n}}[n]$;
   (b) $\mathrm{IC}_1 \star \mathrm{IC}_n \cong \mathrm{IC}_{n+1} \oplus \mathrm{IC}_{n-1}$.
(2) Describe, for any $n \geq 0$, the graded vector space $\mathrm{H}^\bullet(\mathrm{Gr}, \mathrm{IC}_n)$.
(3) Using the results of the preceding questions, show in this special case that the category $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$ is semisimple.
(4) Using the results of the preceding questions, show in this special case that the convolution product of two perverse sheaves in $\mathrm{Perv}_{G(\mathscr{O})}(\mathrm{Gr})$ is a perverse sheaf.

Hints for (1):

- You might use the exercise for Lecture 3. (This exercise was about the affine Grassmannian of GL_2 but, for any connected component of $\mathrm{Gr}_{\mathrm{GL}_2}$, the natural map $\mathrm{Gr}_{\mathrm{GL}_2} \to \mathrm{Gr}_{\mathrm{PGL}_2}$ restricts to an isomorphism from this component to a connected component of $\mathrm{Gr}_{\mathrm{PGL}_2}$.
- You might use the fact that if X is a complex algebraic variety of dimension n such that for any $x \in X$ we have

$$\dim \mathcal{H}^i((\mathrm{IC}_X)_x) = \begin{cases} 1 & \text{if } i = -n; \\ 0 & \text{otherwise,} \end{cases}$$

then $\mathrm{IC}_X = \underline{\mathbb{C}}_X[n]$. (For a proof, see §1.4 in [W. Borho and R. MacPherson, *Partial resolutions of nilpotent varieties*].)

2. Additional exercise

Exercise 2. In the setting of the previous exercise, describe the simple summands of each $\mathrm{IC}_n \star \mathrm{IC}_m$ and their multiplicities.