EXERCISES FOR LECTURE 13

1. Main exercise

Exercise 1. Let X be a variety equipped with a stratification $(X_s)_{s\in\mathscr{S}}$, and assume that every stratum is simply connected. What is the Grothendieck group $K(\mathrm{Perv}_{\mathscr{S}}(X))$?

Exercise 2. Let $\mathcal{A}$ be the category of (not necessarily finite-dimensional) vector spaces over a field $\Bbbk$. Show that $K(\mathcal{A}) = 0$ (!).

Exercise 3. Let G be a connected reductive group, and let $\mathrm{Rep}^{\mathrm{fd}}(G)$ be the category of finite-dimensional representations of G. The tensor product operation makes the Grothendieck group $K(\mathrm{Rep}^{\mathrm{fd}}(G))$ into a ring.

Now let T be a maximal torus; let W be the Weyl group; and let $\mathbf{X}$ be the character lattice of T. Show that there is a ring isomorphism

$$K(\mathrm{Rep}^{\mathrm{fd}}(G)) \cong \mathbb{Z}[\mathbf{X}]^W$$

where the right-hand side is the set of W-invariant elements in the group ring $\mathbb{Z}[\mathbf{X}]$.

2. Additional exercise

Exercise 4. Let $\mathcal{A}$ be the category of finitely generated modules over a polynomial ring $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$. Let $\mathcal{B} \subset \mathcal{A}$ be the full subcategory consisting of modules that are finite-dimensional over $\mathbb{C}$.

(1) Show that $K(\mathcal{A}) \cong \mathbb{Z}$, and that it is generated by the class of the free module $[R]$. (If you are stuck, do the case $n = 1$ first. For general n, you will need to use some form of Hilbert's syzygy theorem.). Under this isomorphism, what is the class of a 1-dimensional module?
(2) What is $K(\mathcal{B})$?
(3) The inclusion functor $\mathcal{B} \to \mathcal{A}$ induces a homomorphism of Grothendieck groups $K(\mathcal{B}) \to K(\mathcal{A})$. Describe this homomorphism explicitly.
(4) Regard $R = \mathbb{C}[x_1, \ldots, x_n]$ as a *graded* ring by setting $\deg x_1 = \cdots = \deg x_n = 1$. Let $\mathcal{A}'$ be the category of finitely-generated graded modules. Show that $K(\mathcal{A}')$ is isomorphic (at least as an abelian group) to $\mathbb{Z}[v, v^{-1}]$.

Exercise 5. Let $R = \mathbb{C}[x]$, and let $\mathcal{A}$ be the category of R-modules that are finite-dimensional over $\mathbb{C}$, and on which x acts nilpotently.

(1) Show that $K(\mathcal{A}) \cong \mathbb{Z}$.
(2) Because tensor product of R-modules is not exact, tensor product does *not* induce a ring structure on $K(\mathcal{A})$. In fact, the "map"

$$K(\mathcal{A}) \times K(\mathcal{A}) \to K(\mathcal{A}) \qquad \text{given by} \qquad ([M],[N]) \mapsto [M \otimes N]$$

is not even well-defined. Give an explicit example showing that it is not well-defined.
(3) This can be fixed using derived functors: the map

$$K(\mathcal{A}) \times K(\mathcal{A}) \to K(\mathcal{A}) \qquad \text{given by} \qquad ([M],[N]) \mapsto \textstyle\sum_{i \geq 0} [\mathrm{Tor}_i(M, N)]$$

is well defined, and equips $K(\mathcal{A})$ with a ring structure. What is this ring structure? (Hint: ummm, you have to be a bit generous with what you think counts as a "ring.")