EXERCISES FOR LECTURE 12

1. Main exercise

Exercise 1. Let $G = GL_2(F)$, where F is a non-Archimedean local field, and consider the trivial unramified principal series representation

$$\mathrm{Fun}(G_F/B_F).$$

Write I for the Iwahori subgroup of G.

(1) Produce a short exact sequence of representations

$$0 \to \mathbb{C} \to \mathrm{Fun}(G_F/B_F) \to \mathrm{St} \to 0.$$

Calculate the dimension of St^I, and deduce from a fact in lecture that St must be irreducible. Calculate the dimension of St^{G_O}.

(2) Show that for any nontrivial character χ of B_F,

$$\mathrm{Hom}(\mathbb{C}, \mathrm{Fun}(G_F/B_F, \chi)) \simeq 0.$$

(3) Recall from lecture that $\mathrm{Fun}(G_F/B_F)^\vee \simeq \mathrm{Fun}(G_F/B_F, \delta)$, where δ denotes the modulus character of B_F. Argue there exists a dual short exact sequence

$$0 \to \mathrm{St}^\vee \to \mathrm{Fun}(G_F/B_F, \delta) \to \mathbb{C} \to 0.$$

Prove this sequence, and hence the original sequence in (1), does not split.

2. Additional exercises

Exercise 2. In lecture, we sketched some aspects of the classification of representations of the affine Hecke algebra. The goal of this exercise is for you to work out an analogous classification for the affine Weyl group, i.e., at $q = 1$.

(1) Identify the group algebra of the affine Weyl group $\mathbb{C}[W_{\mathrm{aff}}]$ with $\mathbb{C}[W] \ltimes \mathcal{O}(\check{T})$, where W denotes the finite Weyl group, and $\mathcal{O}(\check{T})$ denotes the algebra of polynomial functions on the Langlands dual torus.

(2) Identify the center of $\mathbb{C}[W_{\mathrm{aff}}]$ with $\mathcal{O}(\check{T})^W \hookrightarrow \mathcal{O}(\check{T})$. Deduce that central characters for the affine Weyl group are in bijection with semisimple conjugacy classes σ in $\check{G}$.

(3) For a given conjugacy class σ, denote the corresponding central reduction by

$$\mathbb{C}[W_{\mathrm{aff}}]_\sigma \simeq \mathbb{C}[W_{\mathrm{aff}}] \underset{\mathcal{O}(\check{T})^W}{\otimes} \mathbb{C}_\sigma.$$

Classify all the irreducible representations of $\mathbb{C}[W_{\mathrm{aff}}]_\sigma$. *Hint:* fix a representative $\dot{\sigma} \in \check{T}$, and consider the functor

$$\mathbb{C}[W_{\mathrm{aff}}]_\sigma\text{-mod} \to \mathrm{Vect}, \qquad M \mapsto M \underset{\mathcal{O}(\check{T})}{\otimes} \mathbb{C}_{\dot{\sigma}}.$$

What residual symmetry do the obtained vector spaces have?

(4) Using your results above, construct an irreducible representation of W_{aff} of dimension $|W|$.

(5) For an irreducible representation of W_{aff} of dimension $|W|$, calculate its algebra of self extensions. What can you say about Exts between two arbitrary simples L_1, L_2?