EXERCISES FOR LECTURE 14

1. Main exercise

Exercise 1. In this exercise we take $G = \mathrm{PGL}_2$, with the usual choices for B and T. Recall that we have identified $X_*(T)$ with $\mathbb{Z}$, in such a way that n corresponds to the cocharacter

$$t \mapsto \begin{bmatrix} t^n & 0 \\ 0 & 1 \end{bmatrix},$$

and that in this case we have $W = \mathfrak{S}_2 = \mathbb{Z}/2\mathbb{Z}$.

(1) Compute $\ell(w)$ for any $w \in W_{\mathrm{ext}}$. (In particular, you should find that the element $\omega := s \ltimes 1$ has length 0, where s is the nontrivial element in W.)
(2) Check the formula given in the lecture for the length of dominant coweights.
(3) Show that if $w \in W_{\mathrm{ext}}$ has length 0, then multiplication by $\dot{w}$ induces an autoequivalence of $\mathrm{Perv}_I(\mathrm{Fl}_G)$.
(4) Determine the effect of this autoequivalence on standard, costandard, and simple objects.
(5) Describe the order on W_{ext} determined by
$$y \leq w \quad \Leftrightarrow \quad \overline{\mathrm{Fl}_{G,y}} \subset \overline{\mathrm{Fl}_{G,w}}.$$
Hint: You might use the fact that we have a closed embedding $G/B \hookrightarrow \mathrm{Fl}_G$, and consider multiplication by a lift of a length-0 element in W_{ext}.
(6) Show that for any $w \in W_{\mathrm{ext}}$ we have $\mathrm{IC}_w = \underline{\mathbb{C}}_{\overline{\mathrm{Fl}_{G,w}}}[\ell(w)]$.
Hint: You might first try to show a similar claim for simple I-equivariant perverse sheaves on Gr_G. (For that, you might remember that you have described some of these objects in the exercise for Lecture 9, and use the analogue of the property proved in (3).) Then you might use the fact that the functor $\pi^*[1]$ sends simple perverse sheaves to simple perverse sheaves. (This is a general property of pullback under smooth morphisms, but you might try to check this property explicitly in this case.)

2. Additional exercise

Exercise 2. We continue with the setting of the preceding exercise.

(1) Describe the composition factors of standard and costandard perverse sheaves in $\mathrm{Perv}_I(\mathrm{Fl}_G)$.
(2) Describe the Wakimoto perverse sheaves.
(3) One can check[1] that the object $\mathsf{Z}(\mathrm{IC}^1)$ has a three-step filtration as follows:

IC_ω
$\mathrm{IC}_{s\omega} \oplus \mathrm{IC}_{\omega s}$
IC_ω

.

Describe the Wakimoto filtration of this object, and check the formula for the multiplicities.
(4) Show that a convolution product of perverse sheaves in $D^{\mathrm{b}}(I\backslash G(\mathscr{K})/I)$ is not necessarily perverse.

Exercise 3. Here we consider a general group G.

(1) Check the properties of the Wakimoto objects (except for the support) using the stated properties of standard and costandard perverse sheaves, and the formula for the length of dominant coweights.
(2) It follows from standard properties that the Grothendieck group of the triangulated category $D^{\mathrm{b}}(I\backslash G(\mathscr{K})/I)$ identifies with that of the abelian category $\mathrm{Perv}_I(\mathrm{Fl}_G)$. Check that in this Grothendieck group we have $[\Delta_w] = [\nabla_w]$ for any $w \in W_{\mathrm{ext}}$.
(3) Deduce that for $w, y \in W_{\mathrm{ext}}$ we have $[\Delta_w \star^I \Delta_y] = [\Delta_{wy}]$. (This might require some basic properties of the length function that we have not stated in the lecture.)
(4) Show that for any $\lambda \in X_*(T)$ we have $[\mathscr{W}_\lambda] = [\Delta_\lambda]$.
(5) Deduce the claim about the support of Wakimoto objects.

[1] In fact, this computation is equivalent to that done in Exercise 4.6.8 of Pramod's book.