EXERCISES FOR LECTURE 15

1. Main exercise

Exercise 1. This problem deals with the Artin–Schreier local system $\underline{\mathrm{AS}}$ on $\mathbb{A}^1$. (We'll informally treat $\underline{\mathrm{AS}}$ as a sheaf of "complex" vector spaces.)

(1) Regard $\mathbb{A}^1$ as an algebraic group under addition. Show that $\underline{\mathrm{AS}}$ is a multiplicative local system.

(2) Show that $\Gamma(\mathbb{A}^1, \underline{\mathrm{AS}}) = 0$. Also show that $\mathrm{Ext}^1(\underline{\mathbb{C}}, \underline{\mathrm{AS}}) = 0$. Hint: If this Ext^1-group is nonzero, there is a nonsplit extension of $\underline{\mathrm{AS}}$ by the constant sheaf. This local system on $\mathbb{A}^1$ becomes trivial after pullback along the covering map $x \mapsto x^p - x$ (why?), so it corresponds to some indecomposable representation of $\mathbb{Z}/p\mathbb{Z}$. But in terms of representations, there is no such extension.

Artin's vanishing theorem implies that $\mathrm{Ext}^k(\underline{\mathbb{C}}, \underline{\mathrm{AS}}) = 0$ for $k \geq 2$, so in fact $R\Gamma(\underline{\mathrm{AS}}) = 0$.

(3) Let x be a point in $\mathbb{A}^1$. Show that

$$H^k(\mathbb{A}^1 \smallsetminus \{x\}, \underline{\mathrm{AS}}|_{\mathbb{A}^1 \smallsetminus \{x\}}) = \begin{cases} \mathbb{C} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

2. Additional exercise

Exercise 2. This problem deals with the Artin–Schreier local system $\underline{\mathrm{AS}}$. Let $U = \{[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}]\}$ and $U^- = \{[\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}]\}$. Both groups act on $\mathbb{A}^2$ and hence on $\mathbb{P}^1$ in the usual way. Denote their orbits as follows:

$$U\text{-orbits: } \{[1:0]\},\ V = \{[*:1]\} \qquad U^-\text{-orbits: } \{[1:0]\},\ V^- = \{[1:*]\}$$

Since U^- is isomorphic to $\mathbb{A}^1$, we can regard $\underline{\mathrm{AS}}$ as a (multiplicative) local sytem on it.

Let $a_- : U^- \times \mathbb{P}^1 \to \mathbb{P}^1$ be the action map. Define a functor

$$\mathrm{Av} : D^{\mathrm{b}}_U(\mathbb{P}^1) \to D^{\mathrm{b}}_{U^-,\underline{\mathrm{AS}}}(\mathbb{P}^1)$$

by

$$\mathrm{Av}(\mathcal{F}) = (a_-)_*(\underline{\mathrm{AS}} \boxtimes \mathcal{F})[1].$$

Compute the table of stalks for the image under Av of each simple U-equivariant perverse sheaf.