EXERCISES FOR LECTURE 16

1. Main exercise

If you have never worked with line bundles on $\mathbb{P}^1$, you should do this exercise. If you already have experience with this topic, you might quickly browse this exercise and then go to the additional exercise.

Exercise 1. This exercise deals with line bundles on $\mathbb{P}^1$ (the variety of lines in $\mathbb{C}^2$, which is also the flag variety of SL_2).

(1) Denote by (e_1, e_2) the canonical basis of $\mathbb{C}^2$. Let $U_1 \subset \mathbb{P}^1$, resp. $U_2 \subset \mathbb{P}^1$, be the open subvariety consisting of lines spanned by vectors whose coordinate on e_2, resp. e_1, is nonzero. Show that $\mathbb{P}^1 = U_1 \cup U_2$, that we have identifications $U_1 = \mathrm{Spec}(\mathbb{C}[x])$ and $U_2 = \mathrm{Spec}(\mathbb{C}[y])$ such that
$$U_1 \cap U_2 = \mathrm{Spec}(\mathbb{C}[x, x^{-1}]) \subset U_1 \quad \text{and} \quad U_1 \cap U_2 = \mathrm{Spec}(\mathbb{C}[y, y^{-1}]) \subset U_2,$$
and that these open subschemes being identified via the ring isomorphism
$$\mathbb{C}[x, x^{-1}] \xrightarrow{\sim} \mathbb{C}[y, y^{-1}] \quad \text{defined by} \quad P(x) \mapsto P(y^{-1}).$$

(2) If $B_{\mathrm{SL}_2} \subset \mathrm{SL}_2$ is the Borel subgroup of upper triangular matrices, construct an isomorphism $\mathrm{SL}_2/B_{\mathrm{SL}_2} \xrightarrow{\sim} \mathbb{P}^1$, and describe the open subvarieties U_1 and U_2 in these terms.

(3) For any $n \in \mathbb{Z}$, we consider the locally free $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{O}_{\mathbb{P}^1}(n)$ which is the coherent sheaf associated with the free rank-1 $\Bbbk[x]$-module on U_1, the coherent sheaf associated with the free rank-1 $\Bbbk[y]$-module on U_2, with the identification on the intersection sending $P(x)$ to $y^n P(y^{-1})$. Describe the global sections of this sheaf.

(4) Show that for any $n, m \in \mathbb{Z}$ we have $\mathcal{O}_{\mathbb{P}^1}(n) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(m) \cong \mathcal{O}_{\mathbb{P}^1}(n+m)$.

(5) Show that for any $a \in \mathbb{P}^1$ there exists an exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules
$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_a \to 0$$
(where $\mathcal{O}_a$ is the pushforward to $\mathbb{P}^1$ of the structure sheaf of $\{a\}$).

(6) Show that there exists an exact sequence of sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules
$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$
(The pushforward under the diagonal embedding of the pullback of this exact sequence to the Springer resolution is the coherent counterpart of the Wakimoto filtration of the central sheaf associated with the tautological 2-dimensional representation of SL_2, cf. Exercise for Lecture 15.)

(7) Consider the tautological line bundle on $\mathbb{P}^1$, i.e. the variety of pairs (x, L) where $L \subset \mathbb{C}^2$ is a line and $x \in L$ a vector. Determine its sheaf of sections.

(8) Use Čech cohomology to compute the cohomology of the line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$. (If you don't know Čech cohomology, you can e.g. learn about it in Hartshorne's book.)

2. Additional exercise

Exercise 2. In this exercise we work on the coherent side with the group $G^\vee = \mathrm{SL}_2$.

(1) Show that we have identifications

$$\widetilde{\mathcal{N}} = \{(x, L) \in \mathfrak{sl}_2 \times \mathbb{P}^1 \mid x(L) = 0\},$$
$$\widetilde{\mathfrak{g}^\vee} = \{(x, L) \in \mathfrak{sl}_2 \times \mathbb{P}^1 \mid x(L) \subset L\}.$$

(2) Show that the sheaf of sections of $\widetilde{\mathcal{N}}$ is $\mathcal{O}_{\mathbb{P}^1}(-2)$, and that the sheaf of sections $\mathcal{V}$ for $\widetilde{\mathfrak{g}^\vee}$ sits an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{V} \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

(3) Show that the variety $\widetilde{\mathcal{N}} \times_{\mathfrak{g}^\vee} \widetilde{\mathcal{N}}$ has two irreducible components, describe them, and show they are smooth.

(4) Show that the variety $\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}$ has two irreducible components, describe them, and show they are smooth. (*Hint*: You might want to compute things explicitly using the open covering of $\mathbb{P}^1$ considered in the main exercise.)

(5) Consider the natural morphism

$$\pi : \widetilde{\mathcal{N}} \to \mathfrak{g}^\vee.$$

Show that we have $\pi_* \mathcal{O}_{\widetilde{\mathcal{N}}} \cong \mathcal{O}_{\mathcal{N}}$ where $\mathcal{N} \subset \mathfrak{g}^\vee$ is the subvariety of nilpotent elements. (This property is in fact true for any reductive group.)

(6) Denoting by Z_1 and Z_2 the irreducible components of $\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}$ (with an appropriate choice), show that there exist short exact sequences of coherent sheaves

$$0 \to \mathcal{O}_{Z_1} \to \mathcal{O}_{\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}} \to \mathcal{O}_{Z_2} \to 0,$$
$$0 \to \mathcal{O}_{Z_2}(-1, -1) \to \mathcal{O}_{\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}} \to \mathcal{O}_{Z_1} \to 0.$$

(Here, $\mathcal{O}_{Z_2}(-1, -1)$ is the tensor product of $\mathcal{O}_{Z_2}$ with the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)$ to $\widetilde{\mathfrak{g}^\vee} \times_{\mathfrak{g}^\vee} \widetilde{\mathfrak{g}^\vee}$.) (These exact sequences are coherent counterparts of the Δ- and ∇-filtrations of the big tilting object.)